Abstract- In this paper a new technique of spline methods is used for (0, 1, 3) lacunary interpolation by splines of degree six. An existence and uniqueness theorems of the sextic spline function are studied and also error bound.

Index terms- Spline function, error bound, interpolation.

I. INTRODUCTION

During the middle of the twentieth century the study of spline has received considerable attention. According to Schoenberg [11], the interest in spline function is due to the fact that spline functions are a good tool for the numerical approximation of functions on one hand and that they suggest new, challenging and rewarding problems on the other hand. Piecewise linear functions, as well as step functions, have along been important theoretical and practical tools for approximation of functions. A notable exception was the work done by actuarial mathematician which is called “osulatory interpolation” that began soon after Hermit’s work on interpolation. To get more information about a spline function, one can see it in Ahlberg, Nilson and Walsh’s theory [2]. In the last four decades, Birkhoff [3] has renewed this theory on lacunary interpolation or interpolation by splines. Lacunary interpolation by spline appears whenever observation gives scattered or irregular information about a function and its derivatives. The data in the problem of lacunary interpolation has also values of the functions and its derivatives but without Hermite conditions that only consecutive derivative is used at each node.

In recent years, spline function arise in many problems of mathematical physics, such as solving differential equations in viscoelasticity, hydrodynamics. Also, many problems of fracture mechanics, antenna problems in electromagnetic theory, mixed boundary problems in mathematical physics, biology and engineering, see Albayati [1], Jwamer [8], Jwamer and Faridun [9] and Jwamer and Radha [10].

The developed of lacunary spline function by construct the new type of interpolation with boundary conditions, theoretically and error bounds are improved in the present paper. The best error bounds for different types splines interpolation have been obtained by [3-10]. Fawzy and Holail [6] suggested some local methods for solving lacunary interpolation problems using piecewise polynomials with certain continuity properties. The purpose of this paper is to determine the error bounds for six degree spline of the type (0, 1, 3) by using new technique of spline methods.

This paper is organized as follows: In section 2, we construct a spline function of degree six which interpolates the lacunary data (0, 1, 3), some theorems about existence and uniqueness of a sextic spline function are also studied. In section 3, some some theorems about error bounds and convergence analysis of the new technique are proved.

II. CONSTRUCTION OF THE SPLINE INTERPOLATION

Description of the method: Let \((S_m, C^k)\) be the class of spline functions with respect to the set knots \(x_i\). This class consists of piecewise polynomial functions of degree \(m\) which are smoothly connected in the knots, up to the order \(k, (k < m)\). The spline functions will denoted by \(S_i(x)\), where \(i=0,1,...,m\).
Now we are concerned with spline interpolation:

**Theorem 2.1:**

Given \( \Delta : \{ x_i = ih \}_{i=0}^n \) and real numbers \( \{ f_i, f_i', f_i'' \}_{i=0}^n \),
find \( S(x) \) such that:

\[
S(x_i)=f_i, \quad S'(x_i)=f_i' \quad \text{and} \quad S''(x_i)=f_i'', \quad i=0,1,\ldots,n \text{ and } h=\frac{1}{n}.
\] (2.1)

Then the spline approximation has the form

\[
S_d(x)=S_d(x) = \sum_{j=0}^6 \frac{S_{(j)}(x)}{j!} (x-x_k)^j, \quad x \in [x_k, x_{k+1}],
\]

Where \( k=0,1,\ldots,n-1 \). (2.2)

**Proof:** We shall define each of the \( S_{(j)}^k \) explicitly in terms of the data. In particular we choose

\[
S_{(0)}^k = f_k, \quad S_{(1)}^k = f_k', \quad S_{(2)}^k = f''_k, \quad k=0,1,\ldots,n-1.
\] (2.3)

From Taylor’s series expansion of \( f(x) \) and after some derivation with \( S(x) \), for \( k=1, 2, \ldots, n-2 \), we obtain the following:

\[
S_{(6)}^k = \frac{1}{h^3} (f_{k+1} - 3f_{k+1}^\prime + 3f_{k+1}'' - f_{k+1}'''), \quad (2.4)
\]

\[
S_{(5)}^k = \frac{1}{h^2} (f_{k+1}' - 2f_{k+1}'' + f_{k+1}'''') - hS_{(6)}^k, \quad (2.5)
\]

\[
S_{(4)}^k = \frac{1}{2h} (f_{k+1}'' - f_{k-1}''') - \frac{h^2}{6} S_{(6)}^k, \quad (2.6)
\]

and

\[
S_{(3)}^k = \frac{1}{h} (f_{k+1}' - f_k'' - \frac{h^2}{2} f_k''' - \sum_{r=3}^5 \frac{h^r S_{(r+1)}^k}{r!}). \quad (2.7)
\]

For \( k=0 \), we choose

\[
S_{(6)}^0 = S_{(6)}^1, \quad (2.8)
\]

\[
S_{(5)}^0 = S_{(5)}^1 - hS_{(6)}^1, \quad (2.9)
\]

\[
S_{(4)}^0 = S_{(4)}^1 - hS_{(5)}^1 + \frac{h^2}{2} S_{(6)}^1, \quad (2.10)
\]

and

\[
S_{(3)}^0 =
\frac{1}{h} (f_1' - f_0'' - \frac{h^2}{2} f_0''' - \sum_{r=3}^5 \frac{h^r S_{(r+1)}^0}{r!}). \quad (2.11)
\]

Finally for \( k=n-1 \), we take

\[
S_{(j)}^{n-1} (x_k), \quad j=2, 4, 5 \text{ and } 6. \quad (2.12)
\]

Clearly, the function \( S(x) \) defined in (2.2)–(2.12) solves the \((0, 1, 3)\)-interpolation. Moreover, by construction it is clear that \( S(x) \) is a piecewise polynomial of degree six.

The \( S_{(3)}^1 \) have been chosen to make \( S_{(1)}^k \) right continuous, i.e.,

\[
D_x^2 S_k (x_{k+1}) = D_x^2 f_k (x_{k+1}).
\]

While the \( S_{(j)}^1 \) have been chosen to make \( S(x) \) continuous. Thus

\[
S(x) \in C^{0,1,3} [x_0, x_n] = \{ f \in C [x_0, x_n] : D_x^2 f \in C [x_0, x_n] \}
\]

Where \( D_x \) is the right derivative.

Indeed, \( S(x) \) is the unique piecewise polynomial of degree six in \( C^{0,1,3} [x_0, x_n] \cap C^2 [x_{n-2}, x_n] \), satisfying the interpolation conditions (2.1), \( S(x) \) is a special kind of g-spline, we refer to it as a lacunary g-spline.

The proof of theorem 2.1 is completed.

**III. ERROR BOUNDS**

Suppose \( f \in C^2 [x_0, x_n] \), then using the Taylor expansions it is easy to establish the following lemma showing how well the \( S_{(j)}^k \) approximate \( f^{(j)} (x_k) \) in terms of the modulus of continuity \( W(f^{(j)}; h) \) of \( f^{(j)} (x) \)

**Definition**

\( f \in C^2 [x_0, x_n] \) and \( S(x) \) be a spline function which defined in (2.2), then the quantity \( \max |S_{(j)}^k (x) - f^{(j)} (x)| \) for all \( x \) in \([0,1]\) is called error bounds.

**Lemma 3.1:**

For \( j=2, 4, 5 \text{ and } 6 \) we have \( |S_{(j)}^k - f^{(j)} (x_k)| \)

\[ \leq c_j h^{6-j} W(f^{(6)}; h), \quad k=0, 1, \ldots, n-1, \]

Where \( c_j \) is a constant that depends only on \( j \).
Where the constants $C_{kj}$ are given in the following table:

<table>
<thead>
<tr>
<th>$k=0$</th>
<th>$C_{k2}$</th>
<th>$C_{k4}$</th>
<th>$C_{k5}$</th>
<th>$C_{k6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{7}{180}$</td>
<td>1</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{7}{2}$</td>
</tr>
<tr>
<td>$1 \leq k \leq n-2$</td>
<td>$\frac{67}{360}$</td>
<td>$\frac{3}{4}$</td>
<td>19</td>
<td>$\frac{29}{10}$</td>
</tr>
<tr>
<td>$k=n-1$</td>
<td>$\frac{7}{540}$</td>
<td>$\frac{1}{9}$</td>
<td>$\frac{1}{6}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Proof:

For $k=0$, $x \in [0,1]$ and from (2.3)-(2.10) we get

$$S''_n = \frac{1}{360h} (360 \left(f'_1 - f'_0\right) + h^2 (-8f''_3 + 39f''_2 - 114f''_1 - 127f''_0)).$$

Hence

$$S''_n - f''_n = \frac{1}{360h} (-360hf'' + 360 \left(f'_1 - f'_0\right) + h^2 (-8f''_3 + 39f''_2 - 114f''_1 - 127f''_0)).$$

Using the Taylor series expansions for $f$, $f'$, $f''$ about $x_0$ in above equation, we obtain

$$\left|S''_n - f''_n\right| \leq \frac{7}{180} h^4 w(f^{(6)}; h).$$

Thus $C_{k0} = \frac{7}{180}$. Similarly, we can find $C_{k4}$, $C_{k5}$ and $C_{k6}$.

By the same technique in above, using (2.4)-(2.12), changing the interval $[x_0, x_1]$ into $[x_k, x_{k+1}]$ for $k=1, 2, \ldots, n-1$ and using Taylor series expansions for $f$, $f'$, $f''$ about $x_k$, we get $C_{k2}$, $C_{k4}$, $C_{k5}$ and $C_{k6}$.

This completes the proof of the Lemma 3.1.

**Theorem 3.1:**

Let $f \in C^6[0,1]$ and $S_\Delta$ be the unique lacunary g-spline constructed in (2.2)-(2.12). then for all $j=0, 1, 2, \ldots, 6$ we have

$$\left\|f^{(j)} - S^{(j)}_\Delta\right\|_{L_\infty[x_k, x_{k+1}]} \leq c^*_k h^{6-j} w(f^{(6)}; h), \quad k=0, 1, \ldots, n-1$$

Where the constants $c^*_k$ are given in the following table:

<table>
<thead>
<tr>
<th>$K=0$</th>
<th>$c^*_k0$</th>
<th>$c^*_k1$</th>
<th>$c^*_k2$</th>
<th>$c^*_k3$</th>
<th>$c^*_k4$</th>
<th>$c^*_k5$</th>
<th>$c^*_k6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{11}{180}$</td>
<td>$\frac{107}{360}$</td>
<td>$\frac{673}{720}$</td>
<td>$\frac{7}{3}$</td>
<td>$\frac{37}{12}$</td>
<td>5</td>
<td>$\frac{7}{2}$</td>
</tr>
<tr>
<td>$1 \leq k \leq n-2$</td>
<td>$\frac{173}{1200}$</td>
<td>$\frac{373}{900}$</td>
<td>$\frac{719}{720}$</td>
<td>$\frac{131}{60}$</td>
<td>$\frac{41}{10}$</td>
<td>$\frac{24}{5}$</td>
<td>$\frac{29}{10}$</td>
</tr>
<tr>
<td>$K=n-1$</td>
<td>$\frac{1}{72}$</td>
<td>$\frac{101}{2160}$</td>
<td>$\frac{149}{1080}$</td>
<td>$\frac{13}{76}$</td>
<td>$\frac{7}{9}$</td>
<td>$\frac{7}{6}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Proof:

We sketch the proof of the theorem for \( 1 \leq k \leq n-2 \) while for \( k=0, n-1 \), similar procedures, lead to the required results.

Suppose \( k=1, 2, \ldots, n-2 \), and \( x \in [x_k, x_{k+1}] \), then using the Taylor expansion of \( f(x) \), we have

\[
|f(x) - S_A(x)| = |f(x) - S_k(x)| \leq \sum_{j=0}^{5} \left| \frac{f^{(j)}(x_k) - S_k^{(j)} }{j!} \right| h^j + \left| \frac{f^{(6)}(\delta_k) - S_k^{(6)} }{6!} \right| h^6,
\]

where \( x_k < \delta_k < x_{k+1} \), using Lemma 3.1 and the definition of the modulus of continuity of \( f^{(6)}(x) \), we easily obtain the required result. The other results for derivatives can be easily obtained by the same technique.

IV. CONCLUSION

In this paper, we have studied the existence and uniqueness of the sextic spline function that matches function values, first and third derivatives at the knots. Also, the error estimate was derived theoretically. Also, we conclude that this new technique we used in the proving of the two important theorems which are existences and uniqueness and error bounds in the subject of lacunary interpolation by spline function is very easy than other methods used in [2-8].

REFERENCES


