Sequence of Random Integrals in Hilbert Spaces

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Abstract- In this paper we generalize the concepts of integrating of sequences of random variables in nuclear spaces setting which satisfy with their strong duals the following conditions: the reflexivity, completeness and bornological spaces. Assume that there is a continuous bilinear mapping on the nuclear spaces. For an integrable, predictable processes and square integrable martingales, then there exists a process called the sequence of random variables integrals. The Lebesgue space of these integrable processes is studied and convergence theorems are given. Extensions to general locally convex spaces are presented.

Index terms- Random integrals, Bilinear mapping, Nuclear spaces.

I. INTRODUCTION
J. K. Brooks and J. T. Kozinski studied in their paper [1] the existence of a stochastic integral in a nuclear space setting. The nuclear spaces are assumed to have special generalize properties with pertinent concepts and the presentation of the construction appear in [1].

Theorem 1
Let E, F, and G be nuclear spaces which satisfy the special conditions of the hypotheses of Ustunel, and suppose that there is a continuous bilinear mapping of $E \times F$ into G. Assume that

$$X = \bigcup_{j=1}^{\infty} X_j$$

is an F-valued square integrable martingale and $\sigma-$finite. If H is a bounded E-valued predictable process, then there exists a sequence of G-valued processes

$$\left( \int H \, d X_j \right)_t$$

called the sequence of stochastic integrals of H with respect to $X_j$, which is a square integrable martingale. If we further assume that G has a countable basis of seminorms, then the above conclusion holds when H is a predictable E-valued process, which is integrable with respect to X.

This result extends the theory of nuclear stochastic integration of Ustunel [2] in several directions. In [2] it is assumed that F is the strong dual of E and G is the real number field, and furthermore H is assumed to be bounded. [1] develop their theory and modify the vector bilinear integral developed in [3] for Banach spaces. After defining the space $L_{2,G}^p$, G locally convex, the above bilinear integration theory will be applied when we use the property that a complete nuclear space is a projective limit of a family of Hilbert spaces. We will present the underlying sequence integration theory, and apply this to construct the sequence of stochastic integrals.

II. BILINEAR VECTOR INTEGRATION THEORY
We assume that $E, F$ and $G$ are Banach spaces over the reals $\mathbb{R}$, with norms denoted by $\| \cdot \|$. Let $\Sigma$ be a $\sigma$-field of subsets of the set of sequences $T_j$, and assume $m : \bigoplus \to F$ is a $\sigma$-additive measure. We will assume that there is a continuous bilinear mapping $\Phi$ of $E \times F$ into $G$, which, in turn, yields a continuous linear map $\varphi : E \to L(F,G)$, where $L(F,G)$ is the space of bounded linear operators from $F$ into $G$.

The semivariation of $m$ denoted by $\widehat{m}$ is defined on $\Sigma$ as follows:

$$\widehat{m} (A) = \sup \left\{ \sum e_i \, m (A_i) \right\} \quad (1)$$

Where the supremum is extended over all finite collections of elements $e_i$ in the unit ball $E_i$ of $E$ and over all finite disjoint collections of sets $A_i$ in $\Sigma$ which are contained in $A$. We are only interested in the case when $\hat{m} (T_j) < \infty$ for any $j$ in order to develop a sequence of integration theory of E-valued integrands. Sometimes we will write $\widehat{m}$ as $\widehat{m}_{E,G}$. Note that we write $e$ in place of $\varphi (e)$.

We can show that, for each $A \in \bigoplus \hat{m} (A) = \sup \left\{ m_{z_j} \left( A \right) \right\}$, where the supremum is taken over $z_j \in G_{1j}$, the unit ball of the dual of $G'$ of $G$, and $m_{z_j} : \bigoplus \to E'$ is defined by

$$\sum m_{z_j} (A) e = \left( \sum z_j , e m (A) \right)$$

for $e \in E$. The total variation measure of $m_{z_j}$ is denoted by $\| m_{z_j} \|$. Let

$$m_{E,G} = \left\{ \left| m_{z_j} \right| : \sum z_j \in G_1' \right\}$$

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Thus, $m_{E,G}$ is a bounded collection of positive $\sigma$-additive measures. If $c_0 \not\subset G$ (e.g., if $G$ is a Hilbert space), then one can show that $m_{E,G}$ is relatively weakly compact in the Banach space $c_0(\sum)$ consisting of real-valued measures, with total variation norm. In this case, there exists a positive control measure $\lambda$ such that $m_{E,G}$ is uniformly absolutely continuous with respect to $\lambda$. A set $Q \subset T_j$ is $m$-negligible if it is contained in a set $A \in \sum$ such that $m(A) = 0$.

The advantage of modifying the bilinear integration theorem in [3] to the case where the integrand is operator-valued rather than the measure being operator valued will become apparent when the sequences of nuclear stochastic integrals are studied. We are still able to construct the desired Lebesgue space of sequences of integrable functions and establish convergence theorems. We now sketch this theory (see [1]).

Denote by $S = S_E$ the collection of $E$-valued simple functions.

We say that $h_j : T_j \to E$ are measurable if there exists a series of sequences from $S$ which converges pointwise to $h_j$. We define

$$\sum j=1^\infty N(h_j) = \sup \left\{ \sum j=1^\infty |h_j| \right\} m_j \quad (2)$$

where the supremum is taken over $z_j \in G_j'$. Let

$L = L(m_{EG})$ be the collection of all such $h_j$ with

$$\sum j=1^\infty N(h_j) < \infty \quad \text{Then set} \quad l = l(m_{EG}) \quad \text{to be the}$$

closure of $S$ in $L$.

The space $l$ with the series of seminorms $N$ is our Lebesgue space. There are different, but equivalent ways to define

$$\int \sum j=1^\infty h_j \, dm \quad , \quad h_j \in l$$

We select one which yields more information regarding the defining components. If $h_j \in l$, we can show that there exists a determining double sequences $(h_j)_n$ of elements in $S$, that is, the sequences are Cauchy in $l$, and $(h_j)_n$ converges in $m$-measure, namely,

$$\tilde{m}\left(\left| h_j - (h_j)_n \right| > \varepsilon \right) \to 0 \quad \text{for each} \quad \varepsilon > 0$$

Define the integral of $h_j \in S$ in the obvious manner. A determining sequence for $h_j$ has the property that

$$\{ N(h_j) \}_{n} I_{(.)} \in$$

is uniformly absolutely continuous with respect to $\tilde{m}$. Also

$$(h_j)_n \to h_j \quad \text{in} \quad l$$

The set wise limit

$$\int A \sum \left( h_j \right)_n \, dm \quad , \quad A \in \sum$$

exists and defines a $\sigma$-additive measure on $\Sigma$. Denote this limit by $\int \left( h_j \right)_n \, dm$.

This limit is independent of the choice of the determining sequence for $h_j$. We refer to $l$ as the space of integrable sequences of functions.

We have the Vitali theorem:

**Theorem 2**

Let $(h_j)_n$ be sequences of integrable functions. Let $h_j$ be an $E$-valued sequence of measurable functions. Then $h_j \in l$ and

$$(h_j)_n \to h_j \quad \text{in} \quad l \quad \text{if and only if}$$

(i) $(h_j)_n \to h_j \quad \text{in} \quad m$-measure, and

(ii) $\left\{ N(h_j)_n \right\}_{n} \quad \text{is uniformly absolutely continuous} \quad \text{with respect to} \quad \tilde{m}$.

Now we can show the Lebesgue dominated convergence theorem.

**Theorem 3**

Let $g \in l$, and let $(h_j)_n$ be sequences of functions from $l$. If

$$(h_j)_n \to h_j \quad \text{in} \quad m$$

- measure and

$$\left| \left( h_j \right)_n(.) \right| \leq \left| g(.) \right| \quad \text{for each} \quad n$$

then $h_j \in l$ and

$$(h_j)_n \to h_j \quad \text{in} \quad l$$

**Theorem 4**

If $m_{E,G}$ is relatively weakly compact, then $l$ contains the bounded measurable functions.

**III. SEQUENCES OF RANDOM INTEGRALS IN BANACH SPACES**

The stochastic setting is as follows see [5]. Let $(\Omega,M,P)$ be a probability space. $L^2_F(p)$ is a space of $M$-measurable, $E$-valued functions such that

$$\sum j=1^\infty E\left( |f_j|^2 \right) = \int \sum j=1^\infty |f_j|^2 \, dP \quad < \infty$$

endowed with norms

$$\sum j=1^\infty |f_j| = \sum j=1^\infty E\left( |f_j|^2 \right)^{\frac{1}{2}}$$

Assume

$(M_t)_{t \geq 0}$ is a filtration which satisfies the usual conditions. Suppose $X : R_+ \times \Omega \to F$ is a cdlag adapted process (see [1]), with $X^j_t \in L^2_F$ for each $t$. Let $R$ be the ring of subsets of $R_+ \times \Omega$ generated by the predictable rectangles; thus $\sigma(R) = D$ the
predictable \( \sigma \)-field. Let \( m = I_X \) be the additive \( L^2 \)-valued measure first defined on the predictable rectangles by \( m \left((s,t] \times A\right) = I_A \left(X_t^j - X_s^j\right), A \in \mathcal{M}_s \).

and \( m \left(O_A\right) = I_A \times X_o^j, A \in \mathcal{M}_0 \).

We regard \( E \) as being continuously embedded into \( L\left(L^2_F, L^2_G\right) \) in the obvious manner.

The theory of \[ \text{for Banach stochastic integration} \] can be shown to apply in a parallel fashion to this setting, and we state a few pertinent results. If \( c_0 \subset F \), then \( m \) can be extended uniquely to a \( \sigma \)-additive \( L^2_F \)-valued measure if and only if \( m \) is bounded on \( R \). For our purposes ( see \[ \text{[1]} \]), we will be interested only in the case when all the spaces are Hilbert spaces and \( X \) is a square integrable martingale. In this case, \( \hat{m} \in \mathcal{T}_{c_0} \left(R \times \Omega\right) < \infty \).

We can construct the sequence of the stochastic integrals \( \int_t^s H d X^j \), which is a process such that \( \int_0^t H d X^j \in L^2_G \) and this sequence of processes are \( G \)-valued square integrable martingales. If we still denote the extension of \( m \) to \( D \) by \( m \) then \( \int_0^t H d X^j \) is defined to be \( \int_0^t H I_{[0,t]} d m \), where \( H \) is integrable with respect to \( \hat{m} \).

that is, \( H \in \mathcal{L} \left(m_{E,\mathcal{T}_{c_0}}\right) \) and the Hilbert spaces involved in the bilinear theory are \( E, L^2_F, L^2_G \). This integral will be used to define the sequences of stochastic integrals in nuclear spaces (see \[ \text{[1]} \]).

Definition of \( L^2_G \)

Assume that \( \left(T_j, \sum, m\right) \) are measure spaces, \( m \) is real-valued and \( \sigma \)-additive. Let \( G \) be a complete locally convex space, and basis of seminorms, defining the topology of \( G \). The functions \( f_j : T_j \to G \) are measurable if they are pointwise limit of simple \( G \)-valued measurable functions in \( S_G \). For \( r_j \in G \) and \( h_j \) being measurable, let

\[ \sum_{j=1}^{\infty} N_{r_j}(h_j) = \left( \int_{0}^{\infty} r_j(h_j)^2 d m \right)^\frac{1}{2} \]

Let \( \tilde{M}_G \) be the space of sequences of measurable functions \( h_j \) such that \( N_{r_j}(h_j) < \infty \) for each \( r_j \in G \). Then \( \tilde{M}_G \) is a locally convex space with \( \{ N_{r_j} : r_j \in G \} \) being a basis of seminorms. Define \( L^2_G \), the space of integrable functions, to be the closure of \( S_G \) in \( \tilde{M}_G \). It can be shown that \( L^2_G \) is the set of sequences of measurable functions \( h_j \), that have a determining sequences \( \left(h_j\right)_n \subset S_G \), that are, the sequences satisfies for each \( r_j \in G \), \( N_j' \left(h_n - h_{m}^j\right) \to 0 \) as \( n, m \to \infty \).

and for each \( \varepsilon > 0 \) and \( r_j \in G \), we have \( |m| \left(h_n - h_j > \varepsilon\right) \to 0 \) as \( n \to \infty \) in this case,

\[ \int_A h_j d m = \lim \int_A \left(h_n - h_j\right) d m \quad A \in \Sigma \]

is well defined for all determining sequences. The bounded sequences measurable functions are in \( L^2_G \) and theorems 2 and 3 are hold. Moreover, we have the following theorem (see \[ \text{[1]} \]).

Theorem 5. Let \( G \) be a complete locally convex space with a countable basis of seminorms. Then \( L^2_G \) is complete.

Remark

Suppose \( E \) and \( G \) are locally convex spaces with \( E \) and \( G \) denoting their respective bases of defining seminorms. Assume \( F \) is a Hilbert space and \( \Phi : E \times F \to G \) is a continuous bilinear mapping that induces \( \Phi : E \to L\left(F, G\right) \) using the continuity of \( \Phi \), observe that for each \( r_j \in G \), there exists a \( P_j \in E \) such that \( \Phi\left(U_{r_j}, F_1\right) \subset U_{r_j} \), where \( U_{r_j} \) and \( U_{p_j} \) are sequences of closed balls induced by \( r_j \) and \( P_j \) respectively. If we define \( P_j \left(r_j\right) \) to be the infimum over all \( P_j \) for which the above inclusion holds, it turns out that \( P_j \left(r_j\right) \) are seminorms and \( U_{p_j(r_j)} \) are the closed convex balanced hulls of \( U_{r_j} \), where the union is taken over those \( P_j \) in the above infimum. Also \( P_j \left(r_j\right) \left(e\right) = \sup \left| z_j e \right|_p \), where the supremum is taken over \( z_j \in U_{r_j}^{0}\left(z_j e : f_j \to \left\{ z_j, e f_j\right\}, f_j \in F\right) \). Call \( P_j \left(r_j\right) \) the seminorms associated with \( r_j \) and \( \Phi \). Note that \( E\left(U_{p_j(r_j)}\right) \) is isometrically embedded in \( L\left(F, G\left(U_{r_j}\right)\right) \) where \( E\left(U_{p_j(r_j)}\right) \) is Banach space consisting of equivalence
classes modulo $\text{Ker}P_J(r_j)$, completed under the norm inducted by $P_J(r_j); G(U_r)$ is similarly defined.

IV. SQUARE INTEGRABLE MARTINGALES IN NUCLEAR SPACES

Let $(\Omega, M, P)$ be a probability space. Let $F$ denote a nuclear space which is reflexive, complete, bornological, and such that its strong dual $F'$ satisfies the same conditions. We say $F$ satisfies the special conditions. These special conditions are the hypotheses of Ustunel, who established fundamental results for square integrable martingales in this setting. Let $E$ be such a space. Then for $E$ and $E'$ there exist neighborhood bases of zero, $\chi$ and $\chi'$ respectively, such that for each $U_j \in \chi$, the space $E \{ U_j \}$ is a sequence of separable Hilbert spaces over the reals, and its separable dual is identified with the Hilbert spaces $E' \{ U_j' \}$ as defined in [5] where $U_j'$ is the polar of $U_j$. Also, $\{ U_j^0 : U_j \in \chi \}$ and $\{ V_j^0 : V_j \in \chi' \}$ are bases of closed, convex, balanced bounded sets in $E'$ and $E$ respectively. For $U_j \in \chi$, we denote by $K(U_j)$ the continuous canonical map from $E$ onto $E \{ U_j \}$.

If $U_j, V_j \in \chi$ and $V_j \subset U_j$ then $K(U_j, V_j)$ is the canonical mapping of $E \{ V_j \}$ onto $E \{ U_j \}$.

Let $(\Omega, M, P)$ be a probability space with $\{ M_t \}_{t \geq 0}$ being a filtration satisfying the usual conditions. The set $\{ X_{U_j} : U_j \in \chi \}$ is called a projective system of sequences of square integrable martingales if for each $U_j$, we have that $X_{U_j}$ is an $E \{ U_j \}$-valued square integrable martingales, and if whenever $U_j, V_j \in \chi$ and $V_j \subset U_j$, then $K(U_j, V_j)X_{V_j}$ and $X_{U_j}$ are indistinguishable. We also assume $X_{U_j}$ is cadlag for each $U_j$. One says that $X_{U_j}$ has a limit in $E$, if there exists a weakly adapted mapping $\tilde{X}_j$ on $R_+ \times \Omega$ into $E$ such that $K(U_j)\tilde{X}_j$ is a modification of $X_{U_j}$ for each $U_j \in \chi$.

The next theorem is crucial for defining of the sequence of the stochastic integrals. Ustunel [1, section II.4], assumed the existence of a limit in $E$ for $X$. This hypothesis was removed in [7]. [1] stated the theorem and provided a brief sketch of the proof, which uses a technique of Ustunel.

Theorem 6.

Let $X$ be a projective system of square integrable martingales. Then there exists a limit $\tilde{X}_j$ in $E$ of $X$ which is strongly cadlag in $E$, and for which $K(U_j)\tilde{X}_j$ is a modification of $X_{U_j}$ for each $U_j \in \chi$. Moreover, there exists a $\nu_j \in \chi'$ such that $\tilde{X}_j$ takes its values in $E \{ V_j^0 \}$.

Let $m^2$ denote the space of real-valued square integrable martingales. Define the mapping $T_j : E' \rightarrow m^2$ by $T_j(e_j') = \{ e_j', X_{U_j}' \}$ where $U_j$ are chosen in $\chi$ so that $e_j' \in E' \{ U_j^0 \}$. Argue that $T_j$ are well defined and linear.

If $e_j' \rightarrow e_j'$ in $E' \{ U_j^0 \}$ for some $U_j \in \chi$, then:

\[
\sum_{j=1}^{\infty} |T_j(e_j') - T_j(e_j')| \leq \sum_{j=1}^{\infty} \left\| X_{U_j}' \right\|_{E(U_j)} \| e_j' - e_j' \|_{E(U_j)}
\]

hence $\{T_j(e_j')\}$ converges to $\{T_j(e_j')\}$ in $L^2(D) = L^2$, and thus $T_j(e_j') \rightarrow T_j(e_j')$ in $m^2$. Consequently, $T_j$ are continuous on $E' \{ U_j^0 \}$.

Since $E'$ is bornological, $T_j$ are continuous on $E'$. As a result, $T_j : E' \rightarrow m^2$ are nuclear maps of the form:

\[
\sum_{j=1}^{\infty} T_j(e_j') = \sum_{j=1}^{\infty} \lambda_j \langle e_j', e_j' \rangle_j M_j^i
\]

where $\{ \lambda_j \} \in l'$, $\{ e_j' \}$ is equicontinuous in $E$, and $\{ M_j^i \}$ is bounded in $m^2$. Choose $V_j \in \chi'$, $\{ e_j \}$ such that all $e_j \in V_j^0$. Define the process $\tilde{X}_j$ by $\tilde{X}_j = \sum_{i=1}^{\infty} \lambda_i e_j M_j^i$ where we choose $\{ \tilde{X}_j \}$ to be a cadlag version. Then $\tilde{X}_j$ is the desired process.

Now, [1] identify $X$ and $\tilde{X}_j$, and assumed that $X$ takes its values in the Hilbert space $E \{ V_j^0 \}$.

V. CONSTRUCTION OF THE SEQUENCE OF RANDOM INTEGRALS

Assume that $E, F$, and $G$ are nuclear spaces over the reals satisfying the special conditions of the hypotheses of Ustunel. Also assume that $\Phi : E \times F \rightarrow G$ is a continuous bilinear mapping. The neighborhood bases of zero in $E$ and $G$
are denoted by $\chi_E$ and $\chi_G$. Let $X : \mathbb{R}_+ \times \Omega \to F$ be a square integrable martingale. By Theorem 5, we may assume $X$ is Hilbert space valued.

As a result, we may now assume $F$ is a real Hilbert space see [1]. The bilinear map $\Phi$ induces a continuous linear $\phi : E \to L(F,G)$, which in turn induces the continuous linear map $\overline{\phi} : E \to L \left( L^2_F, L^2_G \right)$.

Since $c_0 \not\subset F$, the stochastic measure $m = (I_X)$ first defined on the predictable rectangles can be extended to a $\sigma$-additive measure, still denoted by $m$, $m : P \to L^2_F$.

Note that if $K_1$ and $K_2$ are Hilbert spaces, then $m$ has finite semivariation with respect to each of these embeddings; thus for each $z_j \in (L^2_G)'$, we define

$$m_{z_j} : D \to E' \text{ by } m_{z_j}(A) = \left\langle z_j, m(A) \right\rangle.$$ 

, for $e \in E$. Given any $r_j \in G$, if $z_j \in (U_j)_{N_{r_j}}^0$, then

$$m_{z_j} : D \to E' \left[ U_j^0 \right] = \left[ \left. U_j_{p_j} \right|_{N_{r_j}} \right] = E \left[ U_j_{p_j} \left|_{N_{r_j}} \right. \right]$$

where $p_j \left( N_{r_j} \right)$ are the seminorms associated with $N_{r_j}$ relative to the mapping $E \to L \left( L^2_F, L^2_G \right)$ by

$$\sum_{j=1}^\infty m_{z_j} \left( A \right) = \sum_{j=1}^\infty \left\langle z_j, \left. m(A) \right|_G \right\rangle.$$ 

In fact, $p_j \left( N_{r_j} \right) = p_j \left( r_j \right)$ relative to the mapping $E \to L(F,G)$. Let

$$m_{r_j} = \left\{ \left. \sum_{j=1}^\infty m_{z_j} \right| : z_j \in (U_j)_{N_{r_j}}^0 \right\}$$

Then

$$\sum_{j=1}^\infty \tilde{m}_{r_j} \left( A \right) = \sup \left\{ m_{z_j} \left( A \right) \right\},$$

where the supremum is extended over $z_j \in (U_j)_{N_{r_j}}^0$. Observe that $\sum_{j=1}^\infty m_{r_j}$ is the semivariation of $m$ relative to $L^2_F$.

$$E \left[ \left( U_j \right)_{p_j(r_j)} \right], L^2_G \left[ \left( U_j \right)_{N_{r_j}} \right]$$

which arises from the isometric mapping of $E \left[ \left( U_j \right)_{p_j(r_j)} \right]$ into

$$L \left( L^2_F, L^2_G \left[ \left( U_j \right)_{N_{r_j}} \right] \right).$$

One can show that

$$L^2_G \left[ \left( U_j \right)_{N_{r_j}} \right]$$

are isometrically embedded in the Hilbert space $L^2_G \left[ \left( U_j \right)_{N_{r_j}} \right]$ and, as a result, $m$ has a finite semivariation relative to each of these embeddings; thus

$$\sum_{j=1}^\infty \tilde{m}_{r_j}$$

is finite for each $r_j \in G$, and $\sum_{j=1}^\infty m_{r_j}$ is relatively weakly compact in $ca(D)$.

A process $H : \mathbb{R}_+ \times \Omega \to E$ is a predictable process, or simply measurable, if it is the pointwise limit of processes from $S_E$, the simple predictable $E$-valued processes. For such a measurable process $H$, define, for $r_j \in G$,

$$\sum_{j=1}^\infty n_{r_j}(H) = \sup \left\{ \sum_{j=1}^\infty p_j (r_j) (H) d \left| m_{r_j} \right| \right\}$$

where the supremum is extended over $z_j \in (U_j)_{N_{r_j}}^0$. Let

$$\tilde{M} = \tilde{M} \left( m_{E,l_G} \right)$$

be the space of measurable functions $H$ such that $z_{r_j} \left( H \right) < \infty$ for each $r_j \in G$ then $\tilde{M}$ is a locally convex space containing $S_E$. Let $l = l \left( m_{E,l_G} \right)$ denote the closure of $S_E$ in $\tilde{M}$.

One can show that for each $H \in l$ there exists a determining sequence $(H_n)$ from $S_E$ such that $(H_n)$ is mean Cauchy in $l \left( z_{r_j} \left( H_n - H_m \right) \to 0 \right)$, for each $r_j \in G$, and $\tilde{m}_{r_j} (p (r_j) (H_n - H_m) > \varepsilon) \to 0$ for each $\varepsilon > 0$ and $r_j \in G$.

Now assume $G$ has a countable basis of seminorms, that is, $G$ is now a nuclear Frechet space. Thus there exists a positive measure $\lambda$ such that $\tilde{m}_{r_j} << \lambda$ for each $r_j \in G$. Since $L^2_G$ is complete and, for $H \in S_E$, we have

$$L^2_G$$
\[\sum_{j=1}^{\infty} N_j \left( \int H \, dm \right) \leq z_{\infty} \left( \int H \, dm \right),\]
where the integral is defined in the obvious way, then for general \( H \in L \) with determining sequence \( (H_n) \), we can define
\[\int H \, dm = \lim_{n \to \infty} \int H_n \, dm \in L^2_G,\]
The completeness of \( L^2_G \) ensures that \( \int A H \, dm \) is a function in \( L^2_G \). Define the process
\[\left( \int A H \, dX^j \right)_t = \int_0^t H \, dX^j \mid_{0\to t} \]
by \( \int H \, dm \) called the sequence of stochastic integrals of \( H \) with respect to \( X^j \). We say \( H \) is integrable with respect to \( X^j \) if \( H \in L \). If \( H \in S_E \), one can show that
\[\left( \int H \, dX^j \right)_t = \int_0^t H \, dX^j \]
are \( G \)-valued square integrable martingales. By means of using determining sequences, the general stochastic integral enjoys this property.

Next, assume that \( G \) just satisfies the special conditions (no longer nuclear Frechet).

Let \( H \) be a bounded measurable \( E \)-valued process; (see [1]) hence the range of \( H \) is contained in a closed, bounded, convex, balanced set \( B_1 \), where \( E \left[ B_1 \right] \) is a Hilbert space. By the continuity of \( \Phi \), it follows that \( \Phi \left( B_1, F_1 \right) \) is contained in a bounded set \( B \) having the same properties as \( B_1 \), and \( G \left[ B_1 \right] \) is a Hilbert space.

Algebraically, \( \Phi \) induces \( \Phi_0 : E \left[ B_1 \right] \times F \to G \left[ B \right] \) which is bilinear, and since \( \Phi_0^{-1} \left( \alpha B_1 \right) \subseteq \left( \alpha B_1 \right) \times F \) for every \( \alpha \in \mathbb{R} \), \( \Phi_0 \) is continuous. As a result, this induces a continuous linear map
\[\phi_0 : E \left[ B_1 \right] \to L \left( F, G \left[ B \right] \right),\]
in which turn induces the continuous linear map
\[\phi_0 : E \left[ B_1 \right] \to L \left( \mathbb{L}^2_F, \mathbb{L}^2_G \right).\]
Hence we can define \( m = I_X : D \to \mathbb{L}^2_F \)
as before, which is \( \sigma \)-additive and has finite semivariation relative to \( \phi_0 \),

Since \( H \) is measurable, it is the pointwise limit of functions from \( S_E \), and thus if \( x' \in E' \), \( x' H_n \to x' H \). This implies that \( (x' H)^{-1}(O) \in D \) for any open set \( O \) of the reals. By the reflexivity of \( E \),
\[E \left[ B_1 \right] = E \left[ B_1 \right] = E' \left( B^0_1 \right) = E' \left( B^0_1 \right)\]
since we have chosen \( B_1 = V^0_i \subseteq X' \). Let \( e' \in E' \left( B^0_1 \right) \); then \( e' = \left( x' \right)_{B^0_1} \), and for \( e \in E \left[ B_1 \right] \), it follows that
\[\left( \left( e , e' \right) = \left( \left( x' \right)_{B^0_1} , e \right) = \left( x' , e \right) \right) \]
that is,
\[x' H = e' H \]
As a consequence,
\[H : R_+ \times \Omega \to E \left[ B_1 \right] \]
is weakly measurable, and since \( E \left[ B_1 \right] \) is separable, by the Pettis theorem we conclude that \( H \) is bounded and measurable as an \( E \left[ B_1 \right] \)-valued function.

We now use the integration theory hence there exists a control measure \( \lambda \) in this setting. Since \( c_0 \not\subset G \left( B \right) \), hence it follows that the space of integrable functions, relative to the map \( \phi_0 \) contains the bounded measurable functions.

Thus \( \int H \, dX = H \, dm \in \mathbb{L}^2_{G \mid B} \), and the process \( \left( \int H \, dX^j \right)_t = \int H \, dX^j \left[ 0 \to t \right] \) defines the sequences of stochastic integrals, note that this processes are square integrable martingales. Since the norm on \( G \left( B \right) \) is stronger than any \( r_j \in G \), one can show that \( \mathbb{L}^2_{G \mid B} \) is continuously injected in \( \mathbb{L}^2_G \).

**Remarks**

(i) When we assumed \( G \) was a nuclear Frechet space, we constructed (see[1]) the sequences of stochastic integrals for every \( H \) integrable with respect to \( X \). In particular, if \( H \) is bounded sequences of stochastic integrals agrees with the one constructed by means of using \( \mathbb{L}^2_{G \mid B} \)

(ii) Suppose \( G \) is nuclear Frechet and \( H \) is integrable relative to \( \phi : E \to L \left( \mathbb{L}^2_F, \mathbb{L}^2_G \right) \). For each seminorm \( N_{r_j} \) on \( \mathbb{L}^2_G \), there is a seminorm \( p_j \left(r_j\right) \in \mathcal{E} \) which induces the isometric embedding \( \phi \) of \( E \left( U_j \right) \) into \( L \left( \mathbb{L}^2_F, \mathbb{L}^2_G \right) \).

Thus each
which is a square integrable martingales. The projective system of sequence of square integrable martingales \( \{ H_{p_j(r_j)} \}_{r_j \in G} \) has a limit in \( G \), and this limit is

\[
\left( \int H \, d X \right) := (M)_i, \quad (M)_\infty = \int H \, d X
\]

Since there is a control measure for \( m, \mu_i \), one can show that \( E \left( M_\infty \left| z_i \right. \right) = M^*_i \).

(iii) If \( H_j \) is a sequence of integrals (see [8], [9]) with respect to the sequence of \( X_i \) of an \( F \)-valued square integrals that is

\[
H_j = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} X_i^j
\]

we can show that

\[
\left( \int H_i \, d X_i \right) := (M)_i, \quad (M)_\infty = \int H_i \, d X_i
\]

REFERENCES