Commutative Schemes of Rings and Koszul Dualities to Integral Geometry

I. Verkelov, PhD, Lomonosov Moscow State University*

Abstract — In the following work are established the fundamental equivalences obtained between Moduli\(_n\), and Lie\(_k\), in the \(E_n\) – algebras and \(\infty\) – categories to the formal moduli problems when in these, are applied the Koszul dualities to the deformed categories to establish the derived fundamental equivalences in the schemes of rings and their spectrum. After, this is particularized to the sheaves context whose germs are defined in holomorphic complex bundles to obtain geometries through cycles and co-cycles in integral geometry.

Keywords — Deformed categories, formal moduli problems, Koszul dualities, spectrum.

I. INTRODUCTION

Through of consider in deformation theory that if \(X\), is a moduli space over a field \(k\), of characteristic zero, then a formal neighborhood of any point \(x \in X\), is controlled by a differential graded Lie algebra. This idea was developed by Deligne, Drinfeld and others, but was precise re-written this ideas through of the deformed categories or higher category theory. In the paper [1], we adopt the Grothendieck’s functor [2] of point’s philosophy and we studied the commutative schemes on rings \(R\), to obtaining of the spectrum in duality to be applied in the schemes of categories in derived geometry using functors through the Penrose transform.

Then equivalences of \(\infty\) – categories can be formulated in moduli problems, as the planted in the derived geometry. These equivalencies can be regarded as instances of the Koszul duality such and as was established in the schemes to derived moduli problems\(^1\):

\[
\Phi : \text{Moduli}^\infty_n \rightarrow \text{Alg}_{\text{aug}}^{(n)},
\]

for every \(X \in \text{Moduli}^\infty_n\), and every \(B \in \text{Alg}_{\text{aug}}^{(n)}\), having a canonical homotopy equivalence

\[
\text{Hom}_{\text{Moduli}}(X, \Psi(B)) \rightarrow \text{Hom}_{\text{Alg}_{\text{aug}}}^{(n)}(\Phi(X), B),
\]

Then we have the following result obtained to derived geometry:

Theorem (I. Verkelov, F. Bulnes) 1. 1. Considering the functors \(\Phi\), \(\Psi\), with the properties given by homotopy equivalences and (1.1), we have the following scheme\(^2\):

\[
\text{Hom}_{\text{Moduli}}(X, \text{Spec}(B)) \cong \text{Hom}_{\text{CAlg}(\text{Sp})}^{(n)}(B, S),
\]

Then considering the left and right functors \(F, G\) that appear by the integral transforms that are involved in the \(k\) – modules level. If we consider that these \(k\) – modules are \(D_{G/H}\) –modules then the equivalences given by the Penrose transform [3]

\[
H^0(X, L_{\delta}) \cong \text{ker}(\overline{U}, Q_{\text{BRST}}),
\]

are translated in the equivalences [3, 4]:

\[
M(D_{G/H} – \text{modules} G - \text{equivariants}) \xrightarrow{\phi_{\text{aug}}} M_G(\mathfrak{g}, H),
\]

which are translated in the isomorphism the Hecke categories [3]:

\[
H_{G} \cong M(\mathfrak{g}, Y),
\]

where the Lie algebra \(\mathfrak{g}\), is the loop extension of the loop algebra \(g(t)\).

The study object is precisely these relations between derived categories as applications of the commutative schemes of rings and the Koszul dualities.

\*Correspondence Author (e-mail: verkelov@gmail.com).

\(^1\) Koszul self-duality of the little cubes operad.

\(^2\) The role of \(S\), in the theory of \(\infty\) – categories is analogous to that of the ordinary category of sets in classical category theory. This is a CW-complexes space.
II. NONCOMMUTATIVE GEOMETRY

Let $V_0$, be a finite dimensional vector space over $k$, and let $X_\alpha : \text{CAlg}_{\text{sm}} \to S$ be the formal moduli problem: to every small $k$–algebra $A$, the functor $X_\alpha$, assigns the $\infty$–category of pairs $(V, \alpha)$, where $V$ is a projective $A$–module of rank $n$, and $\alpha : k \wedge A V \to V_0$, is an isomorphism of $k$–vector spaces. We will denote their tangent complex for $T_{X_\alpha}$. This will be formal moduli problem so that $X_\alpha$, assigns to every small $E_n$–algebra $A$, over $k$, the $\infty$–groupoid of pairs $(V, \alpha)$, where $V$, is an $A$–module and $\alpha : k \wedge A V \cong V_0$, is an equivalence. The definition of $X_\alpha$, does not make any use of the commutativity of $A$. Consequently, $X_\alpha$, extends naturally to a functor

$$X_\alpha : \text{CAlg}_{\text{sm}}^{(1)} \to S$$

(2.1)

By definition, the shifted tangent complex of $X_\alpha[-1] \cong \text{End}(V_0)$. If $k$, is the characteristic zero, then by theorem\(^1\) implies that the formal moduli problem $X_\alpha$, can be canonically reconstructed from the vector space $\text{End}(V_0)$, together with their Lie algebra structure. However the formal $E_1$, moduli problem $X_\alpha$, is additional data, since we can evaluate $\hat{X}_\alpha$, on algebras which are not necessarily commutative. Consequently, is natural to expect that the existence of $\hat{X}_\alpha$, to be reflected in some additional structure on the Lie algebra $\text{End}(V_0)$. We observe that $\text{End}(V_0)$, is not merely a Lie algebra: there is an associative product (given by composition) whose commutator gives the Lie bracket on $\text{End}(V_0)$. In fact, this is a general phenomenon as can viewed in the following theorem:

\(^1\)Theorem. Let $k$, be a field of characteristic zero, and let $\text{Moduli}$ denote the full subcategory $\text{Fun}(\text{CAlg}_{\text{sm}}, S)$, spanned by the formal moduli problem over $k$. Then there is an equivalence of $\infty$–categories:

$$\Phi : \text{Moduli} \to \text{Lie}_{k}^{dg},$$

where $\text{Lie}_{k}^{dg}$, denotes the $\infty$–category of differential graded Lie algebra over $k$. Moreover, if $U : \text{Lie}_{k}^{dg} \to \text{Mod}_{k}$, denotes the forgetful functor (which assigns to each differential graded Lie algebra their underlying chain complex), then the composition $U \circ \Phi$, can be identified with the functor $X \mapsto T_X[-1]$.

Theorem 2. 1. Let $k$, be a field, let $n \geq 0$, and let $\text{Moduli}_n$, be the full subcategory of $\text{Fun}(\text{CAlg}_{\text{sm}}, S)$, spanned by the formal $E_n$, moduli problems. Then there exists an equivalence of $\infty$–categories

$$\Phi : \text{Moduli}_n \to \text{Alg}_{\text{aug}}^{(n)}$$

(2.2)

Moreover, if

$$U : \text{Alg}_{\text{aug}}^{(n)} \to \text{Mod}_k,$$

(2.3)

denotes the the forgetful functor

$$A \mapsto m_A,$$

(2.4)

which assigns to each augmented $E_n$–algebra their augmentation ideal, then the composition $U \circ \Phi$, can be identified with the functor $X \mapsto T_X[-n]$.

Proof. To their demonstration see [5, 6, 7].

III. KOSZUL DUALITIES

Fix a field $k$, and an integer $n \geq 0$. The theorem 2. 1, asserts the existence of an equivalence of $\infty$–categories

$$\text{Alg}_{\text{aug}}^{(n)} \cong \text{Moduli}_n \subseteq \text{Fun}(\text{Alg}_{\text{sm}}^{(n)}, S),$$

(3.1)

The appearance of the theory of $E_n$–algebras on both sides of this equivalence is somewhat striking: it is a reflection of the Koszul self-duality of the little $n$–cubes operad [8]. We consider the following definition.

Def. 3. 1. Let $A$, be an $E_n$–algebra over a field $k$. We let $\text{Aug}(A) = \text{Hom}_{\text{Alg}_{\text{gr}}^{(n)}}(A, k) \in S$, denote the space of augmentation on $A$. Suppose that $A$, and $B$, are $E_n$–algebras equipped with augmentation $\varepsilon : A \to k$, and $\varepsilon' : B \to k$. We let $\text{Pair}(A, B) \in S$, denote the homotopy fiber of the mapping spaces

$$\text{Aug}(A \wedge_k B, k) \to \text{Aug}(A, k) \times \text{Aug}(B, k),$$

(3.2)

More informally: $\text{Pair}(A, B) \in S$, is the space of augmentations $\phi : A \wedge_k B \to k$, which are compatible with $\varepsilon$, and $\varepsilon'$.

Let $A$, be an augmented $E_n$–algebra over a field $k$. Then
the construction

\[ (B \in \text{Alg}_\text{aug}^{(n)}) \mapsto (\text{Pair}(A, B) \in \mathcal{S}), \quad (3.3) \]

is a representable functor. In other words, there exists an augmented \( E_n \) - algebra \( D(A) \), and a pairing \( \phi \in \text{Pair}(A, D(A)) \), with the following universal property: for every augmented \( E_n \) - algebra \( B \), over \( k \), composition with \( \phi \), induces a homotopy equivalence

\[ \text{Hom}_{\text{Alg}_\text{aug}^{(n)}}(B, D(A)) \cong \text{Pair}(A, B), \quad (3.4) \]

We well refer to \( D(A) \), as the Koszul dual to \( A \).

By the adjoint functor theorem, (3.4) is equivalent to the assertion that the functor \( B \mapsto \text{Pair}(A, B) \), carries co-limits of spaces.

The construction \( A, B \mapsto \text{Pair}(A, B) \), is symmetric in \( A \), and \( B \). Consequently, for any pair of augmented \( E_n \) - algebras \( A \), and \( B \), we have homotopy equivalences (see Table 1).

\[ \text{Hom}_{\text{Alg}_\text{aug}^{(n)}}(A, D(B)) \cong \text{Pair}(A, B) \cong \text{Pair}(B, A) \cong \text{Hom}_{\text{Alg}_\text{aug}^{(n)}}(B, D(A)).\quad (3.5) \]

Fix a field \( k \), and an integer \( n \geq 0 \). We let

\[ \text{Free} = \text{Mod}_k \rightarrow \text{Alg}_k^{(n)}, \quad (3.6) \]

be a left adjoint to the forgetful functor. That is to say, \( \text{Free} \), assigns to each \( k \) - module spectrum \( V \), the free \( E_n \) - algebra \( \text{Free}(V) \), on \( V \). Note that zero map \( V \rightarrow k \), determines an augmentation on \( \text{Free}(V) \), where we will view to \( \text{Free}(V) \), as an augmented \( E_n \) - algebra.

Let \( A \), be an augmented \( E_1 \) - algebra. According to the second row in the Table 1, to the \( k \) - module \( A \in \text{Mod}_k \)\text{left}.

The Koszul dual of \( A \), can be identified with

\[ \text{End}_A(k) = \text{Hom}_A(k, k) = \text{Hom}_k(k \wedge_A k, k), \quad (3.7) \]

That is, \( D(A) \), can be identified with the \( k \) - linear dual of the bar construction \( BA = k \wedge_A k \). The algebra structure on \( D(A) \), is determined by an associative co-algebra structure on \( k \wedge_A k \), given by

### IV. RESULTS

Considering the role of \( \mathcal{S} \), in the theory of \( \infty \) - categories as the analogous to that of the ordinary category of sets in classical category theory, and the Yoneda embedding defined in [1] for any \( \infty \) - category \( \mathcal{C} \), then in particular to a graded algebra \( H^*(\text{Bun}_G, D^+) \), obtained from a Yoneda embedding, and generated by one copy of \( H^+ \), over \( H^0 \cong \text{Op}_{r_G} \), is had that on a disk that [9]:

**Theorem (E. Frenkel, C. Teleman) 4.1.** The Yoneda algebra \( \text{Ext}^*_{D^+(\text{Bun}_G)}(D^+, D^+) \), is abstractly \( A_{\infty} \) - isomorphic to (the strictly skew-commutative one) \( \text{Ext}^*_{\text{Loc}_{\mathcal{C}}}(\mathcal{O}_{r_G}, \mathcal{O}_{r_G}) \).

Considering a full subcategory of sheaves in \( \mathcal{C} = \text{Coh} (\text{Loc}_{\mathcal{C}}) \), then we have:

\[ H^*(\mathcal{G}([z]), \mathcal{G}; \mathcal{V}_{\text{cat}}) \cong \Omega^*[\text{Op}_{r_G}(D)], \quad (4.1) \]

Then considering an \( A_{\infty} \) - enhancement of (4.1), that is to say, an \( \infty \) - algebra in \( \text{Alg}^{(n)}_{\text{aug}} \), then we can give the

### TABLE I

<table>
<thead>
<tr>
<th>( k ) - module</th>
<th>( \text{Alg}_\text{aug}^{(n)} )</th>
<th>( D(A) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V \in \text{Mod}_k )</td>
<td>( k \oplus V )</td>
<td>( k \oplus V^\vee )</td>
</tr>
<tr>
<td>( A \in \text{Mod}_k )\text{left}</td>
<td>( \text{Hom}<em>{\text{Alg}</em>\text{aug}^{(n)}}(A, D(B)) )</td>
<td>( \text{Hom}<em>{\text{Alg}</em>\text{aug}^{(n)}}(A, D(B)) \circ f )</td>
</tr>
</tbody>
</table>

\[ BA = (k \wedge_A k) \cong k \wedge_A A \wedge_A k \mapsto k \wedge_A k \wedge_A k \cong k \cong \cong BA \wedge_A BA, \quad (3.8) \]
isomorphism
\[ H^0(\mathfrak{g}[z]; \mathbb{V}_{\text{can}}) \cong \ker(\text{Bun}_n^G, \partial), \quad (4.2) \]
which is the Penrose transform to a functor \( \text{Fun}(\mathbb{D}^\text{top}, \mathbb{S}) \). Here \( \text{Bun}_n^G = X \), where \( X \) is the flag variety as the “quantum” version of the construction of an algebra \( \text{sym}\mathbb{T} \).

**Lemma (F. Bulnes, I. Verkelov) 4. 2.** Let \( C \) be a derived category whose functor belongs to the space \( \text{Fun}(\mathbb{D}^\times, \mathbb{Q}) \). Then the cycles and co-cycles in the scheme (1. 4) of the theorem 1. 1, are calculated by the Penrose transform on each ramification \( \partial + d \) of \( \mathcal{O}_{\text{Op}^G} \), having:

\[ \text{End}_G(\mathbb{V}_{\text{can}}) \cong \text{FunOp}_G, \quad (4.3) \]

**Proof.** This a direct consequence of the theorem 4. 2, given in [10] and their corresponding geometrical Langlands correspondence given in relation (4.2) [10].

The following lemma 4. 1, is characterized as a scheme on a deformation problem. Before we consider the following proposition:

**Proposition 4. 1.** Let \( k \) be a field, let \( C \) be a compactly generated \( k \)-linear \( \infty \)-category, and let \( C \in \mathbb{G} \) be an object. Suppose that the following condition is satisfied:

(*) For every compact object \( X \in \mathbb{G} \) the groups \( \text{Ext}^n_C(X, C) \), vanish for \( n >> 0 \). Then \( \text{Def}(C) \), is a formal \( E_1 \)-moduli problem over \( k \).

Under by hypothesis of the proposition 4. 1, and the theorem 6. 1 of [1], asserts the formal \( E_1 \)-moduli problem \( \text{Def}(C) \), is “controlled” by an augmented \( E_1 \)-algebra \( A = \Phi(\text{Def}(C)) \), or equivalently by the augmentation ideal \( \mathfrak{m}_A \) (viewed as non-unital \( E_1 \)-algebra over \( k \)). As an exercise to our readers is possible demonstrate that \( \mathfrak{m}_A \) is equivalent to the endomorphism algebra \( \text{End}_G(C) \), which is not simple coincidence, since precisely we want endomorphisms of type (4. 3) applied in objects of derived categories (as established in the lemma 4. 2).

**Corollary (I. Verkelov) 4. 1.** The moduli problems defined by the equivalences between objects of the algebra \( \text{Alg}_{\text{aug}}^{(n)} \) and the geometrical objects given by the stack moduli defined in the compactly generated \( k \)-linear \( \infty \)-category \( C \), have the following isomorphism:

\[ \text{End}_G(C) \cong \text{Fun}(\mathbb{D}, \mathbb{G}), \quad (4.4) \]

We consider the following proposition.

**Proposition 4. 2.** Let \( A \) be an augmented \( E_n \)-algebra \( k \).

Then there is an equivalence of \( k \)-modules spectra

\[ \mathfrak{m}_n[n] \cong T_{\Psi(A)}, \quad (4.5) \]

**Proof.** We apply again the Koszul duality and the relative details on the inverse limits to obtain \( \text{Spf}^5 \) in the context of “\( \text{CRings} \), \( \text{CAlg}(\mathbb{S}) \)”, to the context of the compactly generated \( k \)-linear \( \infty \)-category \( C \), that is to say, being \( C \in \mathbb{G} \) the pro-objects of \( \text{Pro}(\mathbb{G}) \), can be identified with formal filtered limits \( C = \lim_{\leftarrow} C_n \) of objects \( C \in \mathbb{G} \). Now we consider the functor

\[ \text{Def}(C) : \text{Alg}_{\text{aug}}^{(1)} \rightarrow \mathbb{S}, \quad (4.6) \]

where we want establish a canonical mapping \( \lambda : T_{\text{Def}(C)} \rightarrow \mathfrak{m}_n[-n] \), which must be an equivalence of \( k \)-modules spectra (that is to say objects in derived categories) and satisfying the proposition 4. 2. Then this must be traduced in the context the formal moduli problems on \( \mathbb{G} \), as the application between homomorphisms:

\[ \lambda : \text{Hom}_k(V^\vee, T_{\text{Def}(C)}) \rightarrow \text{Hom}_k(V^\vee, \mathfrak{m}_n[-n]), \quad (4.7) \]

For one side, considering to \( C \), a \( E_n \)-small algebra over \( k \), and the functor

\[ \Psi : \text{Alg}_{\text{aug}}^{(n)} \rightarrow \text{Fun}(\text{Alg}_{\text{aug}}^{(n)}, \mathbb{S}) \quad (4.8) \]

5 If \( \lim_{\leftarrow} A_{\text{aug}} \), is a pro-object of \( \text{Alg}_{\text{aug}}^{(n)} \), we let the functor

\[ \text{Spf}(A) : \text{Alg}_{\text{aug}}^{(n)} \rightarrow \mathbb{S}, \]

as the functor given by the formula

\[ B \mapsto \text{Hom}_{\text{Pro}(\text{Alg}_{\text{aug}}^{(n)})}(A, B) \cong \lim_{\leftarrow} \text{Hom}_{\text{Alg}_{\text{aug}}^{(n)}}(A_{\text{aug}}, B). \]
we have \( \forall C, D \in C \), that

\[
\Psi(C)D = \text{Hom}_{\mathcal{D}}(DD, C), \tag{4.9}
\]

where \( \mathcal{D} \colon (\text{Alg}_{\text{sp}}^{(n)})^{\text{op}} \to \text{Alg}_{\text{sp}}^{(n)} \), is the Koszul duality functor. But by the composition \( j \circ \Psi \), where \( \forall \infty \) – category \( C \), the mapping \( j \), is the Yoneda embedding \([1]\), is had that

\[
\text{Fun}(C', S) \to \text{Fun}(D, O). \tag{4.10}
\]

For other side, considering that \( \text{Def}(C) \), is a formal \( E_1 \) – moduli problem, we denote the representation functor

\[
\text{Def}(C)(D) = \text{Hom}_{\text{Alg}_{\text{sp}}^{(n)}}(C, D) = \Phi(DC)(D), \tag{4.11}
\]

which is an element of the \( k \) – modules spectra, that is to say, of \( \text{Def}(D) \in \text{Fun}(\text{Alg}_{\text{sp}}^{(1)}), O \)\(^6\) and commute with filtered co-limits. Then by the theorem of the page foot 3, and the lemma 4. 2, we have (4. 4).

As special case of the scheme given in (1. 4) and considering to \( C \), a derived category we have the scheme:

\[
\text{Hom}_{\text{Moduli}}(X, \text{Def}(D)) \simeq \text{Hom}_{\text{Alg}(\text{Sp})}(D, O), \tag{4.12}
\]

which can have applications in deformation theory \([12, 13, 14]\), that is to say, considering the functors in the space \( \text{Fun}(D^x, O) \).

V. CONCLUSIONS

The Koszul dualities between formal moduli problems and algebras in the categories context are useful tool to determine equivalences of different object context to design a commutative scheme of the spectrum of rings, when is wanted a formal theory of objects in categories with different characteristics. In particular to objects in \( \infty \) – categories and \( E_n \) – algebras to construct the functor space \( \text{Fun}(D^x, O) \), in deformation theory. From a point of geometrical view this is defined as a formal \( E_1 \) – moduli problem of the cycles and co-cycles that live in the spectrum given in the theorem 1. 1. Using integral transforms the kernel of these transforms are in sheaf \( O_{\text{Op}}^x \), which is studied and discussed in \([1, 9, 10]\). The obtained results are consequences of theorem obtained in \([10]\), and are focused in what happen in the “Rings” \( C\text{Alg}(\text{Sp}) \), when are considered geometrical objects given by the moduli stacks defined in the compactly generated \( k \) – linear \( \infty \) – category \( C \), to that deformation theory through integral transforms.

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References


\(^{6}\)This image is compact in \( \text{Fun}(\text{Alg}_{\text{sp}}^{(n)}, S) \).