Integral Transforms and Opers in the Geometrical Langlands Program

F. Bulnes, PhD, Head of Research Department in Mathematics and Engineering*

Abstract— A problem in derived geometry is determine the cycles and co-cycles that can conform the Langlands correspondence via the Penrose transforms on generalized $D$-modules in moduli stacks defined on adequate holomorphic vector bundles and their possible extension to meromorphic connections, as an example. In this correspondence problem a Zuckerman functor is a factor of the universal functor of derived sheaves of Harish-Chandra which can be worked widely in the Langlands geometrical program to the mirror symmetry in different physical stacks of the Universe.

The cosmological problem that exists is to reduce the number of field equations that are resolute under the same gauge field (Verma modules) and to extend the gauge solutions to other fields using the topological groups symmetries that define their interactions (generalized Verma modules). This extension can be given by a global Langlands correspondence between the Hecke sheaves category on an adequate moduli stack and the holomorphic bundles category with a special connection (Deligne connection).

Keywords — Derived categories, geometrical Langlands program, Langlands correspondence, Penrose transforms.

I. BACKGROUND AND IMPORTANT RESULTS

The determination of a global Langlands correspondence between cycles and co-cycles of different context, one geometrical and other algebraic, is from a point of view of the Langlands program, related with the field ramifications and the holomorphic bundles category that can be determined inside a duality problem.

In this sense and considering adequate extensions of the corresponding Verma modules related to the connections to these ramifications have been considered [1, 2, 3] the corresponding generalizations with an additional geometrical hypothesis in the Zuckerman functor between categories of coherent $D$-modules. Likewise the functor constructed through an integral transforms theory in geometrical analysis comes given for

\[ \Phi + \text{Geometrical hypothesis,} \]

which is established for the geometrical duality of Langlands which affirms that the derived category of coherent sheaves on moduli space $M_{\text{flat}} (L G, C)$, is equivalent to the derived category of $D-$modules on the moduli space $\text{Bun}_G (C)$. Now our moduli space will be $\mathbf{D} (\text{Bun}_G (C))$.

If we want generalize the results foreseen in this context of the derived categories to the ramifications of Langlands, the philosophy to apply is the obtaining of equivalences between the different modules classes establishing the isomorphisms between categories whose Zuckerman functor let be restricted to the sub-category of left equivariant modules that are product of a triangulated sub-category as the given in [4, 5], and for a factor category that comes from certain Zuckerman functor constructed from an image of one derived category using the corresponding generalization of the integral transforms on modules. At least the geometrical aspect will can be solved using the Radon-Penrose transform on modules [20], and the calculated aspect will be a generalization of Penrose transform. To the physical stacks appears, in the nature way (using twisted Hecke categories) the twistor transform linked to the Penrose transform [7, 8, 9].

Through an appropriate induction on the moduli space $M_{\text{flat}} (L G, C)$, in the referent of their context was established the following result in [2] that treats to obtain a geometrical Langlands correspondence using integral geometry method on $D-$modules under dualities of cycles on $G(t)$.

Theorem 1.1 (F. Bulnes) [2]. The stack moduli to $L G-$ holomorphic bundles with regular connections is extendible to irregular connections considering the local Langlands parameters (considered in the micro-local induction of the moduli space $M_{\text{flat}} [L G, C]$) $(\nabla, D^*)$, on

\[ \nabla = \partial_t + A(t), \forall A(t) \in L g(t), \]

where $L g(t)$, is the Lie algebra corresponding to the dual Langlands group $L G(t)$. The space $\mathbf{D}^*$, is the twisted sheaf of differential

*Correspondence to Author (e-mail: francisco.bulnes@tesch.edu.mx).
\( \text{Bun}_{G, y, \infty} \), and using the functors (provided by the \( D \) – module transforms to \((3.1)\) in [2]) we have\(^2\)

\[
H_{G^*, \infty} \cong \mathbb{M}_{K_c}(\mathfrak{g}, Y), \quad (1.2)
\]

under duality of the cycles on \( G(t) \) ( \( B \) – equivariant \( D \) – modules on the flag variety \( G/B \) ) and their geometrical Langlands correspondence is

\[
\mathcal{D}_{\text{BRST}}(\text{Oper}_{G}^{\xi,H}(D_Y)) \cong \mathcal{D}^\times \left( \text{Bun}_G(\Sigma) \right) \quad (1.3)
\]

Note: We consider that \( \mathcal{D}_{\text{BRST}} \) is the derived category on \( D \) – modules of \( Q_{\text{BRST}} \) – operators applied to the geometrical Langlands correspondence to obtain the “quantum” geometrical Langlands correspondence that we want (we want to obtain a differential operators theory from a point of view of BRST-cohomology that includes the theory \( QFT \) (Quantum Field theory), the \( S\text{SU}S \) \( Y \) (Super-symmetry theory) and \( H\text{ST} \) (heterotic string theory)\(^3\)).

Their complete demonstration can be found in [2].

---

operators to our \( \text{Oper}, \) given by \( \text{Loc}_{\Sigma}(\mathbb{D}^\times) \), which is the set of equivalence classes of \( L G \) – bundles with a connection on \( \mathbb{D}^\times \), is a bijection with the set of gauge equivalence classes of the mentioned operators \( \nabla = \partial_t + A(t) \).

\(^1\) Here \( H_{G^*, \infty} \), is the Hecke category that assigns the \( \infty \) – categories of quasi-coherent sheaves on the stack \( \text{Loc}_{\Sigma} \), of \( \Sigma \). Also this Hecke category is \( H_{G, \lambda} \), \( \forall \lambda \in \mathfrak{h}^* \), that is the twisted Hecke category consisting of the natural integral transforms acting on categories of \( \lambda \) – twisted \( D \) – modules of the flag variety \( G/B \) (see the theorem 2.2). This theorem also comes to strengthen the conclusion obtained in the theorem 3.1, and establish a correspondence between the open curves of the moduli stacks of \( G \) – bundles as \( \text{Bun}_{G, y} \), and Hecke algebras.

\( G(t), \) is the loop group that acts naturally on every of the categories \( \mathbb{M}_{K_c}(\mathfrak{g}, Y) = \tilde{\mathbb{M}}_{K_c} \mod \chi, \) in the Frenkel notation \( [8] \).

\(^2\) Remember that this is a correspondence between flat holomorphic \( L G \) – bundles of the worldsheet \( \Sigma, \) (hypersurface) and Hecke eigensheaves \( \forall \lambda \in \mathfrak{h}^* \) on the moduli space \( \text{Bun}_G \), of \( G \) – holomorphic bundles on \( \Sigma, \) where \( G = \text{SL}(n, n + 1) \).

We interpret the Hecke eigensheaves and Hecke operators (elements of the Hecke algebra) of the geometric Langlands program in terms of the correlation functions of purely bosonic local operators in the holomorphic twisted sheaves \( \mathbb{D}_{\lambda, \chi} \), on the complex flag manifold \( \text{SL}(n, n + 1)/B \).

Then by the Recillas conjecture \([4]\) and using the extensions discussed in the section 3, \([2, 5]\) we can to obtain this geometrical correspondence on \( \text{SO}(n, n + 1) \).

After and with the obtaining of some results and identities relative to moduli spaces and their stacks, geometrical and physics; is wanted to extend these geometrical Langlands correspondence to the ambit of meromorphic connections which in the geometrical Langlands program is identified through the global Langlands correspondence between the Hecke sheaves category \( H_{G^*, \infty} \), on an adequate moduli stack and the holomorphic \( \text{Loc}_{G^*}(G) \), bundles category with a special connection (Deligne connection).

Due the generalized geometries that can be obtained through \( D \) – module that can be \( D_{G/H} \) – modules (for the mirror theory induced by the Hitchin moduli space \([4]\)) and considering the Higgs nature of the fields obtained by this procedure \([1, 2]\), was establish and demonstrated in \([9]\) the following result:

**Theorem (F. Bulnes) 1. 2.** Considering \( \text{M}_{\text{Higgs}}(G, C) = T^\gamma_Y \text{Bun}_C(\Sigma), \) and \( \phi_{\mathcal{C}_C(\theta)} = \theta \), defined before, we have

\[
\text{M}(L G, C) = \text{M}_{\text{Higgs}}(L G, C) K^{1/2}, \quad (1.4)
\]

where \( K^{1/2} \), is the square root of the bundle of lines on \( \text{Bun}_G \), corresponding to the critical level.

**Proof.** \([9]\).\(*\)

What happen with the ramifications \( \nabla\), to the case \( H^\ast(L G, \mathcal{C}_C) \otimes P^\ast(K^{1/2}) \otimes L^{[2]} \)?

Geometrical correspondences cannot be determined because \( \nabla\), is not holomorphic?

We can consider some extensions of Moduli stacks and a corresponding induced bundle of lines.

From the Theorem 1, 2, is clear that the ramifications to the part of connection \( \nabla\), must be inside the context of the moduli space \( \text{M}_{\text{Higgs}}(L G, C) \). The induced lines bundle must be one from \( T^\gamma_Y \text{Bun}_C(\Sigma), \) with the condition of that it must be a divisor of holomorphic vector bundle.

To the respect, is had the following theorem \([9]\) in which is considered the results demonstrated in \([9]\), and some ideas of the seminar work given in:

**Theorem (F. Bulnes). 1. 3.** If \( \nabla\), has moduli stack \( \mathcal{C}_\chi = L^{[2]} \), where \( L^{[2]} \), is the sub-bundle of lines

\[
L^{[2]} \cong \mathcal{C}_{\mathcal{C}_C(\theta)} (\otimes \varphi^{\otimes -(n-1)}), \quad (1.5)
\]

where \( \mathcal{C}_C \rightarrow C \times T^\gamma_Y \text{Bun}, \) is simply the cover of \((p_C^\ast V, \phi), \) and hence comes equipped with a natural line.
bundle $\mathcal{L}_\lambda$, such that $\pi_v, \mathcal{L}_\lambda = p_C^* V$, then their generalized Penrose transform \textit{(which is a Penrose-Ward transform)} comes given by

$$H^0(\mathbb{C} G, \Gamma(U, O)) \cong \ker(U, p^* \nabla + \tau(\nabla)), \quad (1.6)$$

\textbf{Proof.} [9]. \noindent

\section{The Oper as Functors in $\mathcal{D}_X$-Schemes}

We want certain considerations that permit us establish the generalizations required to the obtaining of equivalences that can be constructed via a generalized Penrose transforms in the context meromorphic. Possibly this generalization could be in an algebraic context such as is suggested by the Penrose-Ward transform. But is necessary develop more ideas around this.

We give some preliminaries of the geometrical ramifications in the geometrical Langlands program to can investigate this.

If we consider the ramification of the geometrical Langlands program we can to obtain the global Langlands correspondence that should assign a $\mathbb{C} G$-local system $E$, of $X$, (being $X$, a complex variety) tamely ramified at the point $y_1, y_2, \ldots, y_n$, a category $\text{Aut}_E$, of $D$-modules on $\text{Bun}_{G/(y_i)}$, with the eigenvalue $E|_{X - \{y_1, \ldots, y_n\}}$,

$$E \mapsto \text{Aut}_E, \quad (2.1)$$

Suppose that the local system $X - y$, admits the structure of a $\mathbb{C} G$-local system $E$, whose restriction $\chi_y$, to the punctured disc $D_y^\times$, belongs to the subspace $\text{Op}_{nlp}^{D_y}(D_y)$, of nilpotent $\mathbb{C} G$ operas.

For a simple Lie group $G$, the moduli stack $\text{Bun}_{G,y}$, has a realization corresponding to the type [1, 2]

$$\text{Bun}_G \cong \text{D}(\text{B} \setminus \text{G}/\text{B}), \quad (2.2)$$

Then to a $\mathbb{C} G$-local system $E$, on $X$, tamely ramified at $y$, we have the equivalences that are considered in the \textit{theorem} 1.1,

$$D^b(\text{Aut}_E) \cong D^b(\text{QCoh}(\text{Sp}_{\text{Res}(E)})), \quad (2.3)$$

and

$$D^b(\text{Aut}_E^{\text{nilp}}) \cong D^b(\text{QCoh}(\text{Sp}_{\text{Res}(E)}^{\text{DG}})), \quad (2.4)$$

If we consider only the category of $M_{\kappa_c}(\mathfrak{g}, \chi)^0 = \mathfrak{g}_{\kappa_c} - \text{mod}^\rho_{\chi}$, contains irreducible objects $M_{\kappa_{\lambda + p} - \rho}(\chi)$, labeled by the Weyl group $W_{\mathfrak{g}}$, and each object of $\mathfrak{g}_{\kappa_c} - \text{mod}^\rho_{\chi}$, is a direct sum of these irreducible modules \textit{(where these are quotients of the Verma modules)}:

$$M_{\kappa_{\lambda + p} - \rho} = \text{Ind}_{B_{\mathfrak{g}}}^{\mathfrak{g}} \mathbb{Q} C_{\kappa_{\lambda + p} - \rho}, \forall \omega \in W, \quad (2.5)$$

governed by the “control” character corresponding to $\chi$.

Using $\forall \omega \in \mathfrak{z}$, we consider the space $\text{B}_{M}$, a variety of Borel subgroups containing $M$. We observe that it $M$, is regular semisimple then $\text{B}_{M}$, is a set of points which is in bijection with $W$. Therefore from the conjecture mentioned by the isomorphism between derived categories given by (2. 3) and (2. 4) is had that $\text{Aut}_{E_x}^{\text{nilp}}$, is equivalent to the category $\text{QCoh}(B_{M})$, of quasi-coherent sheaves on $B_{M}$, where $M$, is a representative of the conjugacy class of the monodromy of $E_X$.

Finally around $\gamma$, is had that

$$D^b(\text{Aut}_{E_x}^{\text{nilp}}) \cong D^b(\text{QCoh}(B_{M}^{\text{DG}})), \quad (2.6)$$

where this has an obvious generalization to the case of multiple ramification points, where on the right hand side we take the Cartesian product of the varieties $B_{M}^{\text{DG}}$, corresponding to the monodromies. Then it had the conjecture:

\textbf{Conjecture 2.1.} The categories of Hecke eigensheaves, whose eigenvalues are local systems with regular singularities are in terms of categories of quasi-coherent sheaves.

Likewise, the Hecke eigen-sheaves on $\text{Bun}_{G,(y)}$, obtained above via the localizing functor mat be viewed as pull-backs of twisted $D$-modules on $\text{Bun}_{G,y}$, (or more generality extensions of such pull-backs).

Likewise, $\forall \lambda \in \mathfrak{h}^*$, we have the sheaf of twisted differential operator on $\text{Bun}_{G,y}$, acting on a line bundle $\mathcal{L}_\lambda$.

\footnote{Also $B_{M}$, can be viewed as a $\mathbb{D}G$, scheme $\text{Sp}_{\mathfrak{g}}^{\mathbb{D}G}$, in the unipotent case.}
In the case that $\lambda$, is an integral weight, this is a fine bundle as usual, which is constructed on the flag manifold $G/B$, having $\forall \lambda$, the $G-$ equivariant line bundle $L_\lambda = G \times C_\lambda$, on $G/B$.

Thus the line bundle $L_\lambda$, on $\text{Bun}_{G,I}$, is defined in such a way that their restriction to each fiber of the projection $p$, is isomorphic to $L_\lambda$. Then

$$\mathfrak{L}_\lambda = L_\lambda \otimes p^*(K^{1/2}),$$  \hspace{1cm} (2.7)

where $K^{1/2}$, is a square root of the canonical line bundle on $\text{Bun}_G$, corresponding to the critical level.

Recent research have established that even though the line bundle $\mathfrak{L}_\lambda$, does not exist if $\lambda$, is not an integral weight, the corresponding sheaf $D^\lambda_{\mathfrak{k},I}$, of $\mathfrak{L}_\lambda$ twisted differential operators on $\text{Bun}_{G,I}$, is still well defined . Then the corresponding category $g_{\mathfrak{k},y} \mod \phi^0$, has objects

$$M_{w(\lambda + \rho) - \rho}(\chi_y),$$

that we introduced above.

The Cartan sub-algebra $h$, of $g_{\mathfrak{k},y}$, acts on

$$M_{w(\lambda + \rho) - \rho}(\chi_y),$$

semi-simply with the eigenvalues given by the weights of the form $w(\lambda + \rho) - \rho + \mu$, where $\mu$, is an integral weight, thus,

$$M_{w(\lambda + \rho) - \rho}(\chi_y) \otimes C_{w(\lambda + \rho) - \rho},$$

which is $I_y$ equivariant.

Therefore we find that the functor image $\Delta_{\mathfrak{k},y} (M_{w(\lambda + \rho) - \rho}(\chi_y))$, is weakly $H$ equivariant and the corresponding acting of $h$, is given for

$$w(\lambda + \rho) - \rho : h \rightarrow C,$$  \hspace{1cm} (2.8)

Thus $\Delta_{\mathfrak{k},y}^0 (M_{w(\lambda + \rho) - \rho}(\chi_y))$, is the pull-back of a

$$D^\lambda_{\mathfrak{k},I_y}$$

module on $\text{Bun}_{G,y}$.

Thus $D^\lambda_{\mathfrak{k},I_y}$ module is a Hecke eigen-sheaf with eigenvalue $E_{\chi}$, provided thus

$$\chi_y = \chi|_{D^\lambda_{\mathfrak{k},I_y}},$$  \hspace{1cm} (2.9)

where $\chi$, is a regular $\text{Opers}$, on $X \setminus y$.

Then there is an unique Hecke eigen-sheaf on $\text{Bun}_{G,y}$, with eigenvalue $E_{\chi}$, which is a twisted $D-$ module with the twists given by $\mathfrak{L}_{w(\lambda + \rho) - \rho}^{-\tau}$, which could be in the field theory a homogeneous line bundle, image of the twistor transform.

What happens in the cohomological context on the derived categories as for $D_{\mathfrak{k},I_y}^{-\tau}$?

The last image given by $\Delta_{\mathfrak{k},y}^0 (M_{w(\lambda + \rho) - \rho}(\chi_y))$, could be extended as a generalized Verma modules with some additional hypothesis on $\text{Opers}$, and their generalizations of the integral dominant weights.

III. RESULTS: AN PERSPECTIVE THROUGH THE OPERS OF AN TECHNICAL RESULT IN [3]

We consider the following result published a demonstrated in [3], which used a scheme on functoriality of rings in $\text{Loc}_{\mathfrak{k},y}$, considering the connection scheme to the ramified field in the space-time $M$, through the isomorphism scheme given by the Yoneda Algebras:

**Theorem 3.1 (F. Bulnes).** If we consider the category $M_{\mathfrak{k},y}(\mathfrak{g},Y)$, then a scheme of their spectrum $V_{\text{Def}}$, where $Y$, is a Calabi-Yau manifold comes given as:

$$\text{Hom}_{\mathfrak{g}}(X, V_{\text{Def}}) \cong \text{Hom}_{\text{Loc}_{\mathfrak{k},y}}(V_{\text{Def}}, M_{\mathfrak{k},y}(\mathfrak{g}, Y)), $$

(3.1)

We focus in the following higher cohomology in the Yoneda algebra context,

$$H^*(g[[z]], g; V_{\text{critical}}) \cong \Omega^*[\text{Op}_{\mathfrak{k},y}(D)],$$

(3.2)

where on left we have the Lie algebra cohomology with coefficients in the vacuum module at critical level [10]:

$$V_{\text{critical}} := U_{\text{critical}} \otimes g[[z]] C$$

(3.3)

while on the right side $\text{Op}_{\mathfrak{k},y}(D)$, is a variety of $\text{Opers}$, on formal disk.

It’s clear that a scheme that is $D_X$ scheme to derived categories [11] as given inside of scheme (3.1) is the following:
the apparatus of the Penrose transform through their extension, interpreting the invariances under the scheme of “CRings” and $D_X$—schemes.

$$\text{Hom}_{D_X\text{-Schemes}}(X, \text{Op}_{\mathcal{G}}(D)) \cong \text{Hom}_{\text{Loc}_{\mathcal{G}}}(Z(LG, \mathcal{O}_X(X)), (3.4))$$

and applying these to cycles and co-cycles on derived categories of corresponding $Dp$—modules that are quasi-coherent $D_X$—modules we can give the following scheme of the corresponding geometrical Langlands correspondence of the type [2]:

$$D_{\text{twi}}(\text{loc-sys}_{\mathcal{G}}(x)) \cong D(\mathcal{Bun}_G(\Sigma)), \quad (3.6)$$

to know,

$$\text{RHom}_{D_X}(D, D^*) = \text{RHom}_{\text{Loc}_{\mathcal{G}}}(L \mathcal{Bun}_G, L \Phi^*(D)),$$

(3.7)

The scheme given in (3.7) is a $D_X$—scheme, and the equality is demonstrated using the local Serre duality identities on the global section (cycles or co-cycles in each case) of the derived categories $D_{\kappa_x, t}$.

**Corollary 3.1.** A $D_X$—scheme version of the scheme (3.1) is given by (3.7).

**Proof.** Of fact, the use of the Penrose transform as $D$—modules transform [14] on the $Dp$—modules do that the $\text{Op}_{\mathcal{G}}$, can be viewed as affine space for the jet [15, 16, 17]

$$J : \mathcal{O}_X - \text{alg} \to D_X - \text{sch},$$

having $J_X(\Omega_X \otimes T eG)$ over $X$. Then there is a

$$D_X$$—scheme isomorphic to $J_X(\Omega_X \otimes T eG)$. Then the functor image $R \Gamma^H_k$, given for this case as the functor image

$$\Gamma(X, \Omega_X \otimes T eG) \cong D^X \quad [18].$$

Of it is followed the corollary.

IV. CONCLUSIONS

The geometrical Langlands correspondence for $\text{Opers}$, plays an important role starting from the $\text{Loc}_{\mathcal{G}}$, system on $X$, until get a maximal ideal of the corresponding $\infty - \text{algebra}$ whose residue field has a ramification that can be a connection. The Penrose transform inside the geometrical analysis is a tool that help to construct equivalences in moduli problems whose spectrum are derived categories where their $D$—modules are the twisted $D$—modules in a twisted sheaf $D^\times$. The improve of the Penrose transform and their generalizing possibly help to construct a version of a global Langlands correspondence between objects of vector complex bundles and objects of an algebra, considering the ramifications of field, for example in the derived geometry or physics inside maximal ideal. Then considering a scheme on functoriality of rings in $\text{Loc}_{\mathcal{G}}$, takeing the connection scheme to the ramified field in the space-time $M$, a $D_X$—scheme is the obtaining in (3.4) an constituted by (3.7).

Appendix

Likewise, if $X$, be a scheme, for example, the scheme given from the derived categories $D_X$, whose sheaves let $I$, be coherent sheaves of ideals on $X$, then the transformation that we define is the morphism $\pi : \bar{X} \to X$, such that $\pi^{-1}O_{\bar{X}}$, is an invertible sheaf. Here $O_{\bar{X}}$, is the structure of sheaf of $\bar{X}$.

Likewise, morphisms from schemes to affine schemes are completely understood in terms of ring homomorphisms by the following contravariant adjoint pair: For every scheme $\bar{X}$, and every commutative ring $A$, we have a natural equivalence

$$\text{Hom}_{\text{Schemes}}(X, \text{Spec}(A)) \cong \text{Hom}_{\text{CRing}}(A, \mathcal{O}_X(X)), (A.1)$$

since $A$, is an initial object in the category of rings, the category of schemes has as a final object. $\text{Spec}(A)$, means the spectrum of a category of the commutative rings. The character $\text{Spec}$, is the functor “spectrum”.

\[\text{ISSN 2380-2634 (Print) ISSN 2380-2650 (Online)}\]

\[\text{March 2015} \quad \text{http://www.researchpub.org/journal/jm/jm.html}\]
ACKNOWLEDGMENTS

I am very grateful with the state of Mexico government and TESCHA, for the financial support of this work. A special acknowledgment to the financing director B. of Law, Rodolfo Martínez Calles.

References