Two-Temperature Generalized Thermoelastic Infinite Medium with Cylindrical Cavity SubJECTED to Non-Gaussian Laser Beam

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Abstract—The present work is devoted to a study of the induced temperature and stress fields in an elastic infinite medium with cylindrical cavity under the purview of two-temperature thermoelasticity. The medium is considered to be an isotropic homogeneous thermoelastic material. The bounding plane surface of the cavity is loaded thermally by non-Gaussian laser beam with pulse duration of 2 ps. An exact solution of the problem is obtained in Laplace transform space and the inversion of Laplace transforms have been carried numerically. The derived expressions are computed numerically for copper and the results are presented in graphical form.

Index Terms — Thermoelasticity; Two-Temperature Thermoelasticity; Non-Gaussian laser pulse

I. INTRODUCTION

Although thermomechanical phenomena in the majority of practical engineering applications are adequately simulated with the classical Fourier heat conduction equation, there is an important body of problems that require due consideration of thermomechanical coupling: it is appropriate in these cases to apply the generalized theory of thermoelasticity. Serious attention has been paid to the generalized thermoelasticity theories in solving thermoelastic problems in place of the classical uncoupled/coupled theory of thermoelasticity.

The absence of any elasticity term in the heat conduction equation for uncoupled thermoelasticity appears to be unrealistic, since due to the mechanical loading of an elastic body, the strain so produced causes variation in the temperature field. Moreover, the parabolic type of the heat conduction equation results in an infinite velocity of thermal wave propagation, which also contradicts the actual physical phenomena. Introducing the strain-rate term in the uncoupled heat conduction equation, Biot extended the analysis to incorporate coupled thermoelasticity [1]. In this way, although the first shortcoming was over, there remained the parabolic type partial differential equation of heat conduction, which leads to the paradox of infinite velocity of the thermal wave. To eliminate this paradox generalised thermoelasticity theory was developed subsequently. Due to the advancement of pulsed lasers, fast burst nuclear reactors and particle accelerators, etc., which can supply heat pulses with a very fast time-rise Bargmann [2] and Boley [3]; generalised thermoelasticity theory is receiving serious attention. The development of the second sound effect has been nicely reviewed by Chandrasekharaiha [4]. At present mainly two different models of generalised thermoelasticity are being extensively used—one proposed by Lord and Shulman [5] and the other proposed by Green and Lindsay [6]. L-S (Lord and Shulman theory) suggests one relaxation time and according to this theory, only Fourier's heat conduction equation is modified; while G-L (Green and Lindsay theory) suggests two relaxation times and both the energy equation and the equation of motion are modified.

The so-called ultra-short lasers are those with pulse duration ranging from nanoseconds to femtoseconds in general. In the case of ultra-short pulsed laser heating, the high-intensity energy flux and ultra-short duration laser beam, have introduced situations where very large thermal gradients or an ultra-high heating speed may exist on the boundaries Sun et al. [7]. In such cases, as pointed out by many investigators, the classical Fourier model, which leads to an infinite propagation speed of the thermal energy, is no longer valid Tzou, [8], [9]. The non-Fourier effect of heat conduction takes into account the effect of mean free time (thermal relaxation time) in the energy carrier's collision process, which can eliminate this contradiction. Wang and Xu have studied the stress wave induced by nanoseconds, picoseconds, and femtoseconds laser pulses in a semi-infinite solid [10]. The solution takes into account the non-Fourier effect in heat conduction and the coupling effect between temperature and strain rate. It is known that characteristic elastic waveforms are generated when a pulsed laser irradiates a metal surface.

The two temperatures theory of thermoelasticity was introduced by Gurtin and Williams [11], Chen and Gurtin [12], and Chen et. al. [13], [14], in which the classical Clausius-Duhem inequality was replaced by another one depending on two temperatures; the conductive temperature \( \varphi \) and the thermodynamic temperature \( T \), the first

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is due to the thermal processes, and the second is due to the mechanical processes inherent between the particles and the layers of elastic material, this theory was also investigated by Iesan [15].

The two-temperature model was underrated and unnoticed for many years thereafter. Only in the last decade has the theory been noticed, developed in many works, and find its applications mainly in the problems in which the discontinuities of stresses have no physical interpretations. Among the authors who contribute to develop this theory, Quintanilla studied existence, structural stability, convergence and spatial behavior for this theory [16]. Youssef introduced the generalized Fourier law to the field equations of the two-temperature theory of thermoelasticity and proved the uniqueness of solution for homogeneous isotropic material [17], Purj and Jordan studied the propagation of harmonic plane waves, recently [18]. Magaña and Quintanilla have studied the uniqueness and growth solutions for the model proposed by Youssef [17].

The present work is devoted to a study of the induced temperature and stress fields in an elastic infinite medium with cylindrical cavity under the purview of two-temperature thermoelasticity. The medium is considered to be an isotropic homogeneous thermoelastic material. The bounding plane surface of the cavity is loaded thermally by non-Gaussian laser beam with pulse duration of 2 ps. An exact solution of the problem is obtained in Laplace transform space and the inversion of Laplace transforms have been carried numerically. The derived expressions are computed numerically for copper and the results are presented in graphical form.

II. THE GOVERNING EQUATIONS

Let us consider a perfectly conducting elastic infinite body with cylindrical cavity occupies the region \( R \leq r < \infty \) of an isotropic homogeneous medium whose state can be expressed in terms of the space variable \( r \) and the time variable \( t \) such that all of the field functions vanish at infinity. We use the cylindrical system of coordinates \((r, \psi, z)\) with the \( z\)-axis lying along the axis of the cylinder.

Due to symmetry, the problem is one-dimensional with all the functions considered depending on the radial distance \( r \) and the time \( t \). It is assumed that there is no external forces act on the medium.

Thus the field equations in cylindrical one dimensional case can be put as in Youssef [20]:

\[
(\lambda + 2\mu)\frac{\partial^2 e}{\partial r^2} - \frac{\gamma}{r} \frac{\partial T}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2},
\]

\[
\nabla^2 \varphi = \frac{\rho C_e}{K} \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \theta + \frac{T_o}{K} \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) e^r - \frac{\rho L_0}{K} \left( 1 + \tau_0 \frac{\partial}{\partial t} \right) Q,
\]

\[
\varphi - T = a \nabla^2 \varphi,
\]

\[
\sigma_{rr} = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma (T - T_o),
\]

\[
\sigma_{\psi\psi} = 2\mu \frac{\partial u}{\partial r} + \lambda e - \gamma (T - T_o),
\]

\[
\sigma_{r\psi} = \sigma_{\psi r} = \sigma_{rz} = \sigma_{zr} = 0,
\]

\[
e = \frac{1}{r} \frac{\partial}{\partial r} \left( ru \right).
\]

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \), \( \lambda, \mu \) Lame’s constants, \( \rho \) density, \( C_e \) specific heat at constant strain, \( \alpha_r \) coefficient of linear thermal expansion, \( \lambda = (3\lambda + 2\mu) \alpha_r \), \( t \) is the time, \( T \) is the temperature, \( T_o \) is the reference temperature, \( \theta = (T - T_o) \) is the thermo-dynamical temperature increment such that \( |\theta|/T_o << 1 \), \( \varphi \) is the heat conductive temperature, \( \sigma_{ij}, i, j = r, \psi, z \) are the components of stress tensor, \( e \) is the cubic dilatation, \( u \) is the displacement, \( K \) is the thermal conductivity, \( \tau_0 \) is the relaxation time, \( a \) is non-negative parameter(two-temperature parameter), and \( Q \) is the heat source per unit mass.

III. THE NON-GAUSSIAN LASER PULSE

Consider a beam with initial temperature distribution \( T(x, z, 0) = T_o \). From time \( t = 0 \) its upper surface \( (z = 0) \) is irradiated uniformly by a laser pulse with non-Gaussian temporal profile as in [7]:

\[
L(t) = \frac{L_0}{t_p} \exp \left( \frac{t}{t_p} \right),
\]

where \( t_p = 2 \) ps is a characteristic time of the laser-pulse (the time duration of a laser pulse), \( L_0 \) is the laser intensity which is defined as the total energy carried by a laser pulse per unit area of the laser beam, see Fig. 1 [7].

The conduction heat transfer in the medium can be modeled as a cylindrical one-dimensional problem with an energy source \( Q(z, t) \):

\[
Q(z, t) = \frac{R_s}{\delta} \exp \left( \frac{z}{\delta} \right) L(t) = \frac{R_s L_0}{\delta t_p^2} \exp \left( \frac{z - t}{\delta t_p} \right),
\]

where \( \delta \) is the absorption depth of heating energy and \( Ra \) is the surface reflectivity .

Without loss of generality, we will consider the case when \( z = 0 \), then we have

\[
\nabla^2 \varphi = \frac{\rho C_e}{K} \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \theta + \frac{T_o}{K} \left( \frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) e^r - \frac{\rho L_0}{K} \left( 1 + \tau_0 \frac{\partial}{\partial t} \right) Q.
\]

Equations (15)-(20) assume the form (where the primes are suppressed for simplicity)
\[ \frac{\partial e}{\partial r} - b \frac{\partial \theta}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \]

The last equation could be change to the following form

\[ \nabla^2 e - b \nabla^2 \theta = \frac{\partial^2 e}{\partial t^2}, \]

\[ \nabla^2 \varphi = \left( \frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2} \right) \varphi + e_i \left( \frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2} \right) e - e_i \left( 1 + \tau \frac{\partial}{\partial t} \right) \exp \left( -\frac{t}{t_p} \right), \]

where \( e_i = \sqrt{\lambda + 2\mu}/\rho \) is longitudinal wave speed, \( \eta = \rho C_v/K \) is the thermal viscosity, \( e_i = \gamma / \rho C_v \) is the dimensionless mechanical coupling constant, \( \alpha = \gamma T_\infty / \mu \) is the dimensionless thermoelastic coupling constant, \( \omega = \alpha C^2 \eta^2 \) is the dimensionless two-temperature parameter, \( \beta = \left( \lambda + 2\mu / \mu \right)^{\frac{1}{2}} \), \( b = \alpha / \beta^2 \) and \( e_i = (R_n L_o) / (C_m T_o t_p^2 \delta) \).

**IV. THE SOLUTION IN THE LAPLACE TRANSFORM DOMAIN**

We use the Laplace transform of both sides of the last equations defined as:

\[ \mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} \, dt. \]

Hence, we obtain

\[ \nabla^2 \varphi = s^2 \varphi + b \nabla^2 \overline{\varphi}, \]

and

\[ \nabla^2 \overline{\varphi} = h \overline{\varphi} + e_i h \overline{\varphi} - F(s), \]

where \( F(s) = e_i \left( 1 + \tau \gamma s \right) \) and \( h = \left( s + \tau \gamma s^2 \right) \)

where all the state functions in equations (20)-(26) have zero initial value and an over bar symbol denotes its Laplace transform and \( s \) denotes the Laplace transform parameter.

Eliminating \( \overline{\varphi} \) from the equations (21) and (22), we get

\[ \left( \nabla^2 - \alpha_1 \right) \overline{\varphi} = \alpha_2 \overline{\varphi} - \alpha_1 F(s), \]

and

\[ \overline{\varphi} = \left( 1 - \omega \alpha_1 \right) \overline{\varphi} - \omega \alpha_2 \overline{\varphi} + \frac{\omega \alpha_1}{h} F(s), \]

where \( \alpha_1 = \frac{h}{1 + \omega \alpha_1} \) and \( \alpha_2 = e_i \alpha_1 \).

By using equations (27) and (28) into equation (20), we get

\[ \left( \nabla^2 - \alpha_1 \right) \overline{\varphi} = \alpha_4 \overline{\varphi} - \frac{\alpha_1}{h} F(s), \]

where \( \alpha_4 = \frac{s^2 + \alpha_2 \left( 1 - \omega \alpha_1 \right)}{1 + \omega \alpha_2 b} \), \( \alpha_4 = \frac{\alpha_2 \left( 1 - \omega \alpha_1 \right)}{1 + \omega \alpha_2 b} \).

Eliminating \( \overline{\varphi} \) from equations (27) and (29), we obtain

\[ \left[ \nabla^4 - \left( \alpha_1 + \alpha_4 \right) \nabla^2 + \left( \alpha_1 \alpha_2 - \alpha_2 \alpha_4 \right) \right] \overline{\varphi} = \alpha_5 F(s), \]

where \( \alpha_5 = \frac{\left( \alpha_1, \alpha_2 - \alpha_2, \alpha_4 \right)}{h} \).

In a similar manner, we can show that \( \overline{\varphi} \) satisfies the equation

\[ \nabla^2 - \left( \alpha_1, \alpha_1 \right) \nabla^2 + \left( \alpha_1 \alpha_2 - \alpha_2 \alpha_4 \right) \overline{\varphi} = 0. \]

For bounded solutions at infinity, equations (30) and (31) can be written in the form

\[ \overline{\varphi} = \frac{F(s)}{h} + \sum_{n=1}^{\infty} A_n \left( p_n^2 - \alpha_1 \right) K_0(p_n r), \]

and

\[ \overline{\varphi} = \sum_{n=1}^{\infty} B_n K_0(p_n r), \]

where \( K_0(.) \) is the modified Bessel function of the second kind of order zero. \( A_1, A_2, B_1 \) and \( B_2 \) are all parameters depending on the parameter \( s \) of the Laplace transform.

\( p_1^2 \) and \( p_2^2 \) are the roots of the characteristic equation

\[ p^4 - \left( \alpha_1 + \alpha_2 \right) p^3 + \left( \alpha_1 \alpha_2 - \alpha_2 \alpha_3 \right) p^2 + \left( \alpha_1 \alpha_3 - \alpha_2 \alpha_4 \right) = 0. \]
\( B_i = \alpha_i A_i, \quad i = 1, 2. \) (35)

Substituting from equation (35) into equation (33), we get
\[
\bar{\sigma} = \alpha \sum_{i=1}^{2} A_i K_0(p_r). \tag{36}
\]

Substituting from equation (36) into equation (26), we obtain
\[
\bar{u} = -\alpha \sum_{i=1}^{2} A_i K_i(p_r), \tag{37}
\]
where \( K_i(.) \) is the modified Bessel function of the second kind of order one.

In deriving equation (37), we have used the following well-known relation of the Bessel function
\[
\int z K_0(z) \, dz = -z K_1(z).
\]

We can get the dynamical temperature in the following form
\[
\bar{\theta} = \frac{F(s)}{h} + \sum_{i=1}^{2} \theta_i A_i K_0(p_r), \tag{38}
\]
where
\[
\theta_i = (1 - \omega \alpha_i)(p_i^2 - \alpha_i) - \omega \alpha_i \alpha_4, \quad i = 1, 2.
\]

Finally, substituting from equations (32), (36) and (37) into equations (23)-(25), we obtain the stress components in the form
\[
\sigma_{rr} = -\frac{\alpha F(s)}{h} + \sum_{i=1}^{2} A_i \left[ (\beta^2 - 2) \alpha_4 - \alpha \theta_i \right] K_0(p_r), \tag{39}
\]
\[
\sigma_{\varphi \varphi} = -\frac{\alpha F(s)}{h} + \sum_{i=1}^{2} A_i \left[ (\beta^2 - 2) \alpha_4 - \alpha \theta_i \right] K_0(p_r), \tag{40}
\]
\[
\sigma_{zz} = -\frac{\alpha F(s)}{h} + \sum_{i=1}^{2} \left[ (\beta^2 - 2) \alpha_4 - \alpha \theta_i \right] A_i K_0(p_r). \tag{41}
\]

To complete the solution in the Laplace transform space, we will consider the medium described above is quiescent and the bounding plane of the cavity \((r = R)\) has not any thermal or mechanical loading:
\[
\varphi(R, t) = 0. \tag{42}
\]

After using Laplace transform, we have
\[
\bar{\varphi}(R, s) = 0. \tag{43}
\]
and
\[
\sigma_r(R, t) = 0. \tag{44}
\]

After using Laplace transform, we have
\[
\sigma_r(R, s) = 0. \tag{45}
\]

Apply the last two conditions, we obtain
\[
\sum_{i=1}^{2} A_i (p_i^2 - \alpha_i) K_0(p, R) = \frac{F(s)}{h}, \tag{46}
\]
and
\[
\sum_{i=1}^{2} A_i \left[ (\beta^2 - 2) \alpha_4 - \alpha \theta_i \right] K_0(p, R) + \frac{2 \alpha_4}{R p_i} K_i(p, R) = \frac{\alpha F(s)}{h}. \tag{47}
\]

After solving the last system of equations, we get
\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix} = \begin{bmatrix}
l_{11} \\
l_{12}
\end{bmatrix}^{-1} \begin{bmatrix}
-1 \\
\alpha
\end{bmatrix} \frac{F(s)}{h}, \tag{48}
\]
which gives
\[
A_1 = \frac{F(s)(\alpha l_{12} - l_{22})}{h(l_{11} l_{22} - l_{12}^2)} \quad \text{and} \quad A_2 = \frac{F(s)(l_{11} - \alpha l_{12})}{h(l_{11} l_{22} - l_{12}^2)},
\]
where
\[
l_{11} = (p_1^2 - \alpha_i) K_0(p, R), \\
l_{12} = (p_2^2 - \alpha_i) K_0(p, R), \\
l_{21} = (\beta^2 - \alpha_4 - \alpha \theta_i) K_0(p, R) + \frac{2 \alpha_4}{R p_i} K_i(p, R), \\
l_{22} = (\beta^2 - \alpha_4 - \alpha \theta_2) K_0(p, R) + \frac{2 \alpha_4}{R p_i} K_i(p, R).
\]

Finally, we obtain the solution in the forms:
\[
\bar{\theta} = \frac{F(s)}{h} \left[ 1 + \frac{1}{h(l_{11} l_{22} - l_{12}^2)} \left( (\alpha l_{12} - l_{22}) \left( p_1^2 - \alpha_i \right) K_0(p, R) + \left( l_{11} - \alpha l_{12} \right) \left( p_2^2 - \alpha_i \right) K_0(p, R) \right) \right], \tag{49}
\]
\[
\bar{u} = \frac{F(s)}{h} \left[ 1 + \frac{1}{h(l_{11} l_{22} - l_{12}^2)} \left( (l_{11} - \alpha l_{12}) \left( p_1^2 - \alpha_i \right) K_0(p, R) + \left( l_{11} - \alpha l_{12} \right) \left( p_2^2 - \alpha_i \right) K_0(p, R) \right) \right], \tag{50}
\]
\[
\bar{\sigma}_{rr} = \frac{F(s)}{h} \left[ \frac{\alpha F(s)}{h(l_{11} l_{22} - l_{12}^2)} \left( (\alpha l_{12} - l_{22}) K_0(p, R) \right) + \frac{2 \alpha_4}{R p_i} K_i(p, R) \right], \tag{51}
\]
\[
\bar{\sigma}_{\varphi \varphi} = \frac{F(s)}{h} \left[ \frac{\alpha F(s)}{h(l_{11} l_{22} - l_{12}^2)} \left( (\alpha l_{12} - l_{22}) K_0(p, R) \right) \right], \tag{52}
\]
\[
\bar{u} = -\frac{\alpha F(s)}{h p_i p_2 (l_{11} l_{22} - l_{12}^2)} \left( p_1 (l_{11} - \alpha l_{12}) K_1(p, R) \right) \tag{53}
\]

V. NUMERICAL INVERSION OF LAPLACE TRANSFORM

In order to determine the conductive and thermal temperature, displacement and stress distributions in the time domain, the Riemann-sum approximation method is used to obtain the numerical results. In this method, any function in Laplace domain can be inverted to the time domain as
\[
f(t) = \frac{e^{\alpha t}}{t} \left[ \frac{1}{2} \bar{f}(\kappa) + Re \sum_{n=0}^{N} (-1)^n \bar{f} \left( \kappa + i \pi n \right) \right], \tag{54}
\]
where \( Re \) is the real part and \( i \) is imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of \( \kappa \) satisfies the relation \( \kappa t \approx 4.7 \) [9].
VI. NUMERICAL RESULTS AND DISCUSSION

With a view to illustrating the analytical procedure presented earlier, we now consider a numerical example for which computational results are given. For this purpose, copper is taken as the thermoelastic material for which we take the following values of the different physical constants [20]:

\[ K = 386 \text{ kg m}^{-1} \text{s}^{-2} \text{ K}^{-1}, \alpha = 1.78 \times 10^{-5} \text{ K}^{-1}, \rho = 8954 \text{ kg m}^{-3}, \]
\[ C_e = 383.1 \text{ m}^{-2} \text{ K}^{-1} \text{s}^{-2}, T_0 = 293 \text{ K}, \mu = 3.86 \times 10^6 \text{ kg m}^{-1} \text{s}^{-2}, \]
\[ \lambda = 7.76 \times 10^6 \text{ kg m}^{-1} \text{s}^{-2}, \beta^2 = 4, \quad R = 1.0, \quad \tau_0 = 0.02, \quad t = 0.2. \]

From the above values, we get the non-dimensional values of the problem as:

\[ b = 0.01041, \alpha = 0.0417232, \epsilon_1 = 1.618, \epsilon_2 = 103. \]

In Figs. 2-6, the distributions of conductive temperature, thermo-dynamical temperature, the radial stress, the displacement, and the deformation at different value of two-temperature parameter; the solid lines when \( \omega = 0.0 \) give one-temperature model (Lord-Shulman), the dotted lines when \( \omega = 0.1 \) give two-temperature model (Youssef model).

According to the Figs. 2-6, the two-temperature parameter plays vital role on the speed of the wave propagation of all the studied fields.

REFERENCES


![Fig. 2: The conductive temperature distribution](image-url)
Fig. 3: The thermo-dynamical temperature distribution

Fig. 4: The radial stress distribution

Fig. 5: The displacement distribution

Fig. 6: The deformation distribution