Abstract — The aim of this work is to present numerical data on the velocities of torsional wave propagation in a poroelastic hollow circular cylinder. The frequency equations have been derived in the form of a determinant involving Bessel functions. The values of the characteristic numbers governing the velocities of the first five modes for various geometries of the circular cylinder are numerically computed. The phase velocities of the disturbances for the different wave numbers corresponding to the first five modes are calculated. The results indicate that the effects of porosity and coupling parameters are very pronounced.

Keywords: Torsional wave, Poroelastic hollow cylinder

I. INTRODUCTION

The study of torsional wave propagation is of importance, both from theoretical and practical considerations. These investigations have many theoretical and practical applications in several fields like geophysics, seismology, acoustics, optics, and plasma physics. Poroelasticity is a theory that models the interaction of deformation and fluid flow in a fluid-saturated porous medium. The deformation of the medium influences the flow of the fluid and vice versa. Many materials encountered in civil, geophysical and biomechanical engineering can be considered as porous media consisting of an assemblage of solid particles and pore space. The pore space may be filled with air (dry medium), a fluid (saturated medium) or both (unsaturated medium). The theory of linear isotropic poroelasticity was introduced by Biot [2]. The derivation of constitutive relations followed the generalized Hookes law. Based on symmetry arguments, it was demonstrated that there existed four independent material constants for isotropic poroelasticity, two more than that for elasticity. Biot [3,4,5] studied the propagation of the plane harmonic seismic waves in fluid saturated porous solids. Using this theory, Abbas [1] studied the radial vibrations of a poroelastic hollow cylinder. Deresiewicz [7] and Jones [8] have studied the propagation of free surface waves in a saturated poroelastic half-space while Paul [9,10], Philippacopoulos [11,12] and Schanz and Cheng [13] have considered transient waves.

The object of this paper is to study the torsional wave propagation in a poroelastic hollow cylinder. The values of the characteristic numbers governing the velocities of the first five modes for various geometries of the circular cylinder are numerically computed. The phase velocities of the disturbances for the different wave numbers corresponding to the first five modes are calculated. The variations of phase velocities with different values of the porosity and coupling parameters are plotted.

II. FORMULATION OF THE PROBLEM

Let us consider an infinite circularly cylindrically bar of poroelastic material with its longitudinal axis coinciding with the z-axis of the cylindrical coordinate system \((r, \theta, z)\). Inner and outer radial \(a\) and \(b\) respectively. Six equations that govern the wave propagation in poroelastic bodies according to [2] are,

\[
\frac{\partial^2}{\partial t^2} (\rho_1 \frac{\partial}{\partial r} + \rho_2 \frac{\partial}{\partial r}) = (A + 2N) \nabla e - N \nabla \cdot \nabla \times \frac{\partial}{\partial t} + Q \nabla e, \quad (1)
\]

\[
\frac{\partial^2}{\partial t^2} (\rho_2 \frac{\partial}{\partial r} + \rho_3 \frac{\partial}{\partial r}) = \nabla [Qe + R\epsilon],
\]

where \(e\) and \(\epsilon\) are the dilatations of the solid and fluid phases, respectively,

\[
e = \text{div}(u), \quad \epsilon = \text{div}(v),
\]

\[
\text{div}(u) = \text{div}(v).
\]
with \( \ddot{u} = (u_x, u_y, u_z) \) and \( \ddot{v} = (v_x, v_y, v_z) \) as the displacement vectors of solid and fluid respectively. The \( \rho \)'s are mass coefficients such that the sums \( \rho_1 + \rho_2 \) and \( \rho_1 + \rho_2 \) represent the mass of solid and the mass of fluid per unit volume of the bulk material, respectively. The coefficient \( \rho_2 \) is a mass coupling parameter between the fluid and solid phases. The material coefficients \( N \) and \( A \) are related to the solid phase and correspond roughly to lamine coefficients \( \lambda \) and \( \mu \) of the theory of elasticity. The coefficients \( Q \) and \( R \) represent the parameters relating the stress in the solid \( \tau \) and the excess fluid pressure \( S \), and given by

\[
N = \mu, \quad A = \lambda + M (\alpha - \beta)^2, \quad Q = \beta (\alpha - \beta) M, \quad R = \beta^2 M,
\]

where \( \lambda \) and \( \mu \) are the Lame constants under condition of constant pore pressure, \( \beta \) the porosity, \( \rho \) the grain density, \( \rho_f \) the fluid density, \( \alpha \) and \( M \) are the elastic coefficients related to the coefficient of fluid content \( \gamma \), unjacketed compressibility \( \delta \) and jacketed incompressibility

\[
K = \left( \lambda + \frac{2}{3} \mu \right)
\]

\[
\alpha = 1 - \delta K, \quad M = \left( \gamma + \delta - \delta^2 K \right)^{-1}.
\]

Considering torsional vibrations of the medium, the only non-zero displacements are \( u_x = u_0(\tau, z, t), v_x = v_0(\tau, z, t) \).

All the stress components except \( \tau_{xx} \) and \( \tau_{xy} \) vanish. These two components are given by

\[
\tau_{xx} = 2\mu e_{xx}, \quad \tau_{xy} = 2\mu e_{xy}.
\]

The strain-mechanical displacement relations are

\[
e_{xx} = \frac{1}{2} \left( \frac{\partial u_x}{\partial \tau} - \frac{u_x}{\tau} \right), \quad e_{xy} = \frac{1}{2} \frac{\partial u_y}{\partial \tau}.
\]

The non-trivial stress equations of motion without body forces are

\[
P_{\tau_1} \frac{\partial^2 u_x}{\partial \tau^2} + P_{\tau_2} \frac{\partial^2 v_x}{\partial \tau^2} = \frac{\partial \tau_{xx}}{\partial \tau} + \frac{\partial \tau_{xx}}{\partial \tau} + 2 \frac{\partial \tau_{xy}}{\partial \tau},
\]

\[
P_{\tau_1} \frac{\partial^2 u_y}{\partial \tau^2} + P_{\tau_2} \frac{\partial^2 v_y}{\partial \tau^2} = 0.
\]

For harmonic torsional vibrations, the equations (6) may be satisfied by taking \( u_x \) and \( v_x \) in the form

\[
u_x = u(\tau) e^{i(\omega \tau + \phi)}, \quad v_x = v(\tau) e^{i(\omega \tau + \phi)}.
\]

Substituting (7) into (6) yield

\[
n \frac{\partial^2 u}{\partial \tau^2} + \frac{1}{n} \frac{\partial u}{\partial \tau} + \left( m^2 - \frac{1}{n^2} \right) = 0,
\]

where \( q \) denotes \( \frac{2\pi}{l} \) (\( l \) is the wave length) and \( w \) denotes 2\( \pi \) times frequency of the wave and

\[
m^2 = \frac{w^2 (\rho_1 \rho_{22} - \rho_{12}^2)}{\mu \rho_{22}} - q^2.
\]

III. SOLUTION OF THE PROBLEM

Suppose that the given body is a hollow circular cylinder with \( r = a \) and \( r = b \) as inner and outer boundaries. The solution of (8) may be taken as

\[
u_x(\tau, z, t) = (A_1(\tau) + B_1(\tau)) e^{i(\omega \tau + \phi_1)},
\]

\[
v_x(\tau, z, t) = -D_1(\tau) e^{i(\omega \tau + \phi_1)}.
\]

If the boundaries \( r = a \) and \( r = b \) of the cylinder which separate the body from the free-space are stress free, then the required boundary conditions are

\[
\tau_{xx} = 0 \text{ on } r = a \text{ and } r = b.
\]

From the above boundary conditions and using the recurrence relations for Bessel functions, we get

\[
AX_{11} + BX_{12} = 0,
\]

\[
AX_{21} + BX_{22} = 0,
\]

where we have set

\[
X_{11} = \eta h J_0(\eta h) - 2J_1(\eta h),
\]

\[
X_{12} = \eta h J_1(\eta h) - 2J_0(\eta h),
\]

\[
X_{21} = \eta h J_0(\eta h) - 2J_1(\eta h),
\]

\[
X_{22} = \eta h J_1(\eta h) - 2J_0(\eta h),
\]

with \( h = \frac{a}{b} \eta = mb \).

The condition for a non-trivial solution of (13) is that the determinant of the coefficients of these integration constants must vanish, which leads to the following frequency equation:

\[
\Delta = \begin{vmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{vmatrix} = 0.
\]

The roots of this equation give the values of the characteristic numbers \( \eta \) for the torsional oscillations of a poroelastic hollow circular cylinder. Knowledge of the \( \eta \) values satisfying (15) allows computation of the wave velocities for different modes of motion since

\[
c = \frac{w}{q} \left( \frac{\mu \rho_{22}}{\rho_1 \rho_{22} - \rho_{12}^2} \right) \left( 1 + \frac{\eta^2}{k^2} \right)^{1/2},
\]

where \( k = \frac{2\pi b}{l} \) is the wave number. Also, the non-dimensional phase velocity \( c^* \) is given by

\[
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\[ c^* = \frac{\rho_{22}^c}{\mu} = \frac{\rho_{22}^c}{(\rho_{12}^c + \rho_{12}^n)} \sqrt{1 + \eta^2} \, \left(\frac{k^2}{\eta^2} \right). \]  

(17)

IV. NUMERICAL RESULTS

The roots of (15) give the values of the characteristic numbers governing the velocity of the torsional vibrations of a poroelastic hollow circular cylinder. These roots, which correspond to various modes, are numerically calculated by a "method of halving the interval" also known as the bisection method or Bolzano method [6]. Computational work is carried out for the following data [14,15]:

\[ \mu = 0.922 \times 10^{11} \text{ dynes/cm}^2, \quad \beta = 0.23, \]
\[ \rho_s = 2.66 \text{ gm/cm}^3, \rho_f = 1 \text{ gm/cm}^3. \]

Figure 1 shows the oscillatory nature of the determinant \( \Delta \) as a function of \( \eta \), for different values of \( h = 0.4, 0.6, 0.8 \). The amplitudes of these oscillations seem to increase with increasing frequency and with increasing values of \( h \). The first five modes of \( \eta \) are shown in figure 2 for different values of \( h \). Moreover, two types of graphs are drawn (figures 3, 4, 5 and 6) representing phase velocity \( c^* \) for different wave number with the coupling parameter and the porosity, for the first five modes and third mode respectively. The numerical results of the phase velocity \( c^* \) for different wave number corresponding to the first five modes, taking the characteristic numbers \( \eta \) at \( h = 0.9 \) are shown in figures 3-4. Figures 5-6 shows the variation of phase velocity \( c^* \) for different wave number corresponding to the third mode, taking the characteristic numbers \( \eta \) at \( h = 0.5 \). It is obvious from figure 2 that the first five modes of characteristic numbers increase with the decrease of the thickness. We have plotted on figures 3 the phase velocity as functions of wave number for the value \( \beta = 0.23 \) when neglect the coupling parameter \( \rho_{12} = 0.0 \) and at \( \rho_{12} = -0.05 \).

Figure 4 shows that the variation of phase velocity as function of wave number for two different value of porosity \( (\beta = 0.20, \beta = 0.23) \) at \( \rho_{12} = -0.02 \). Figures 5-6 shows that the variation of the phase velocity with wave number, for different values of \( \rho_{12} \) and \( \beta \) from which it is seen that the effects of coupling parameters and porosity are very pronounced.
Fig. 4. Variation of phase velocity $c^*$ for different values of wave number $k$ to the first five modes of characteristic numbers $\eta$.

Fig. 5. Variation of phase velocity $c^*$ for different values of wave number $k$ to third mode of characteristic numbers $\eta$.

Fig. 6. Variation of phase velocity $c^*$ for different values of wave number $k$ to third mode of characteristic numbers $\eta$.

REFERENCES


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