A Two-dimensional Wave Propagation in a Poroelastic Infinite Circular Cylinder

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Abstract — The present paper investigates the transient wave propagation in a two-dimensional poroelastic infinite circular cylinder. Using Biot’s equations, derived in his consolidation theory, four coupled governing differential equations for the propagation of harmonic longitudinal waves in circularly cylindrical bars of poroelastic material are derived. Numerical values for stresses, pressure and displacement at various points of the cylindrical medium are computed for appropriate material constants and the results are displayed graphically.

Keywords: Torsional wave, Poroelastic hollow cylinder

I. INTRODUCTION

Poroelasticity is a theory that models the interaction of deformation and fluid flow in a fluid-saturated porous medium. The deformation of the medium influences the flow of the fluid and vice versa. Many materials encountered in civil, geophysical and biomechanical engineering can be considered as porous media consisting of an assemblage of solid particles and pore space. The pore space may be filled with air (dry medium), a fluid (saturated medium) or both (unsaturated medium). The theory of linear isotropic poroelasticity was introduced by Biot [1]. The derivation of constitutive relations followed the generalized Hookes law. Based on symmetry arguments, it was demonstrated that there existed four independent material constants for isotropic poroelasticity, two more than that for elasticity. Biot [2, 3, 4] studied the propagation of the plane harmonic seismic waves in fluid saturated porous solids. Using this theory, Deresiewicz [5] and Jones [7] have studied the propagation of free surface waves in a saturated poroelastic half-space while Paul [8, 9], Philippapoulos [10, 11] and Schanz and Cheng [12] have considered transient waves. Another approach to describe the dynamic behavior of porous media, is known as the theory of Porous Media, Ehlers [6] is based on the theory of mixtures and derived from the well known methods of continuum mechanics.

The object of this paper is to study the transient wave propagation in a two-dimensional poroelastic infinite circular cylinder. Using Biot’s equations, derived in his consolidation theory, four coupled governing differential equations for the propagation of harmonic longitudinal waves in circularly cylindrical bars of poroelastic material are derived. Numerical values for stresses, pressure and displacement at various points of the cylindrical medium are computed for appropriate material constants and the results are displayed graphically.

II. FORMULATION OF THE PROBLEM

Let us consider an infinite circularly cylindrically bar of poroelastic material with its longitudinal axis coinciding with the $z$-axis of the cylindrical coordinate system $(r, \theta, z)$. The radius of the cross section is $a$. Six equations that govern the wave propagation in poroelastic bodies according to [1] are,

\[
\frac{\partial^2}{\partial t^2}(\rho_1 \ddot{u} + \rho_2 \ddot{v}) = (A + 2N)\ddot{\bar{V}}v - N\ddot{\bar{V}}\wedge \ddot{\bar{V}} + \bar{Q}\ddot{\bar{V}}e, \\
\frac{\partial^2}{\partial t^2}(\rho_1 \ddot{u} + \rho_2 \ddot{v}) = \bar{V}[Qe + Re],
\]

where $e$ and $\varepsilon$ are the dilatations of the solid and fluid phases, respectively.
with $\vec{u}$ and $\vec{v}$ as the displacement vectors of solid and fluid respectively. The $\rho$’s are mass coefficients such that the sums $\rho_{11} + \rho_{12}$ and $\rho_{12} + \rho_{22}$ represent the mass of solid and the mass of fluid per unit volume of the bulk material, respectively. The coefficient $\rho_{12}$ is a mass coupling parameter between the fluid and solid phases. The material coefficients $N$ and $A$ are related to the solid phase and correspond roughly to lame coefficients $\lambda$ and $\mu$ of the theory of elasticity. The coefficients $Q$ and $R$ represent the parameters relating the stress in the solid $\tau_s$ and the excess fluid pressure $S$; and given by

$$N = \mu, \ A = \lambda + M (\alpha - \beta)^2, \ Q = \beta (\alpha - \beta) M, \ R = \beta^2 M,$$

where $\lambda$ and $\mu$ are the Lame constants under condition of constant pore pressure, $\beta$ the porosity, $\rho_i$ the grain density, $\rho_f$ the fluid density, $\alpha$ and $M$ are the elastic coefficients related to the coefficient of fluid content; $\gamma$ unjacketed compressibility $\delta$ and jacketed incompressibility $K = (\lambda + 2\mu)/3$ by

$$\alpha = 1 - \delta K, \ M = (\gamma + \delta - \delta^2 K)^{-1}.$$

In the axisymmetric case under investigation, equations (1) become

$$\begin{align*}
\frac{\partial^2}{\partial t^2}(\rho_{11} u_{rr} + \rho_{12} v_{rz}) &= (A + 2N) \frac{\partial e}{\partial r} + N \frac{\partial}{\partial z} \left[ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right] + Q e_r, \\
\frac{\partial^2}{\partial t^2}(\rho_{12} u_{rz} + \rho_{22} v_{zz}) &= (A + 2N) \frac{\partial e}{\partial z} - N \frac{\partial}{\partial r} \left[ \frac{\partial u_r}{\partial r} - \frac{\partial u_z}{\partial z} \right] + R e_z,
\end{align*}$$

$$\begin{align*}
\frac{\partial^2}{\partial t^2}(\rho u_{rr} + \rho v_{rz}) &= Q \frac{\partial e}{\partial r} + R \frac{\partial e}{\partial r}, \\
\frac{\partial^2}{\partial t^2}(\rho u_{rz} + \rho v_{zz}) &= Q \frac{\partial e}{\partial z} + R \frac{\partial e}{\partial z},\tag{2a}
\end{align*}$$

where

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}, \quad \varepsilon = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z},$$

with $u_r, v_r, u_z, v_z$ as radial and longitudinal displacements.

The constitutive equations of the poroelastic medium in its axially symmetric form are now

$$\begin{align*}
\tau_{rr} &= 2Ne_{rr} + A e + Q e, \\
\tau_{r\theta} &= 2Ne_{r\theta} + A e + Q e, \\
\tau_{\theta\theta} &= 2Ne_{\theta\theta} + A e + Q e, \\
\tau_{zz} &= 2Ne_{zz} + A e + Q e, \\
S &= Q e + Re,
\end{align*}$$

(3a)

where $S$ is a quantity proportional to the fluid pressure and the remaining notation is standard. The strain components are given in terms of the displacements by

$$\begin{align*}
e_{rr} &= \frac{\partial u_r}{\partial r}, \\
e_{r\theta} &= \frac{1}{2} \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial r} \right), \\
e_{\theta\theta} &= \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta} \right), \\
e_{zz} &= \frac{\partial u_z}{\partial z},
\end{align*}$$

(3b)

### III. Solution of the Problem

We now consider the propagation of an infinite train of sinusoidal waves along a circular cylinder of finite extent such that the displacement at each point is a simple harmonic function of $z$ and $t$, so that

$$u_r = f_j(r) \cos(\omega t + qz), \quad u_z = f_j(r) \sin(\omega t + qz), \quad v_r = f_j(r) \cos(\omega t + qz), \quad v_z = f_j(r) \sin(\omega t + qz),$$

(4)

with $\omega$ as the wave frequency, $q = \frac{2\pi}{l}$ is the wavenumber, $l$ is the wavelength and $f_j(r)$, $j = 1, 2, 3, 4$, as functions of the radius only to be determined later. With (4) in mind equations (3) may now be written in the form

$$\begin{align*}
-w^2 (\rho_{11} f_1 + \rho_{12} f_3) &= (A + 2N) \frac{\partial e}{\partial r} + Nq \left[ \frac{\partial f_1}{\partial r} + \frac{df_3}{dr} \right] + Q e_r, \\
-w^2 (\rho_{12} f_3 + \rho_{22} f_1) &= -(A + 2N) qe + N \frac{d}{dr} \left( qf_3 + r \frac{df_1}{dr} \right) - qe_r, \\
Q \frac{de}{dr} + R \frac{de}{dr} &= -w^2 (\rho_{11} f_1 + \rho_{12} f_3), \\
Q e_r + Re &= w^2 (\rho_{12} f_3 + \rho_{22} f_1), \\
e = \frac{1}{r} \frac{d}{dr} (rf_1) + qf_2, \\
\varepsilon &= \frac{1}{r} \frac{d}{dr} (rf_3) + qf_4.
\end{align*}$$

(5a)

(5b)

(5c)

(5d)
From above equations we get

\[
\frac{d^2e}{dr^2} + \frac{1}{r} \frac{de}{dr} + \alpha_1 e + \alpha_2 e = 0,
\]

\[
\frac{d^2e}{dr^2} + \frac{1}{r} \frac{de}{dr} + \alpha_3 e + \alpha_4 e = 0,
\]

(6)

with the notation,

\[
\alpha_1 = \frac{w^2}{c_1^2} - q^2, \quad \alpha_2 = \frac{w^2}{c_2^2}, \quad \alpha_3 = \frac{w^2}{c_3^2} - q^2, \quad \alpha_4 = \frac{w^2}{c_4^2},
\]

\[
c_i^2 = \frac{Q^2 - R(A + 2N)}{Qp_{32} - Rp_{12}}, \quad c_i^2 = \frac{Q^2 - R(A + 2N)}{Qp_{22} - Rp_{12}},
\]

\[
c_i^2 = \frac{R(A + 2N) - Q^2}{(A + 2N)p_{22} - Qp_{12}}, \quad c_i^2 = \frac{R(A + 2N) - Q^2}{(A + 2N)p_{12} - Qp_{11}}.
\]

(7)

Since, from (6)

\[
\varepsilon = -\frac{1}{\alpha_2} \left( \frac{d^2e}{dr^2} + \frac{1}{r} \frac{de}{dr} + \alpha_1 e \right).
\]

(8)

Equations (6) are decoupled using simple manipulations to yield,

\[
\left[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) + m_1^2 \right] \left[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) + m_2^2 \right] e = 0,
\]

(9)

where \( m_1^2 \) and \( m_2^2 \) are the roots of the following algebraic equation

\[
m^4 - m^2 (\alpha_1 + \alpha_5) + \alpha_1 \alpha_5 - \alpha_2 \alpha_4 = 0.
\]

(10)

The general solutions of equation (10) assuming \( e \) to be finite as \( r \to 0 \) may be expressed as

\[
e = A_1 J_0 (m_1 r) + A_2 J_0 (m_2 r),
\]

(11a)

where \( A_1 \) and \( A_2 \) are arbitrary constants and \( J_0 \) is Bessel’s function of the first kind of order zero.

From (11a) in (8) we get

\[
\varepsilon = B_1 J_0 (m_1 r) + B_2 J_0 (m_2 r),
\]

(11b)

where

\[
B_i = \frac{m_i^2 - \alpha_i}{\alpha_i m_i}, \quad i = 1, 2.
\]

(11c)

With the notation

\[
n^2 = \frac{w^2}{c_1^2} - q^2, \quad c_i^2 = \frac{\rho_{22} N}{\rho_{22} \rho_{11} - \rho_{12}^2},
\]

\[
S_1 = \frac{m_1}{N} \left[ (A + N) A_1 + QB_1 - \frac{\rho_{12}}{\rho_{22}} (QA_1 + RB_1) \right],
\]

\[
S_2 = \frac{m_2}{N} \left[ (A + N) A_2 + QB_2 - \frac{\rho_{12}}{\rho_{22}} (QA_2 + RB_2) \right].
\]

(12)

The complementary solution of equation (11a) is

\[
f_i = C_i J_i (nr) + C_2 Y_i (nr),
\]

(13a)

where \( J_i \) and \( Y_i \) are Bessel’s functions of the first and the second kind of order one, \( C_1 \) and \( C_2 \) are arbitrary constants.

In order to obtain particular solution of equation (11a), let us consider \( C_1 \) and \( C_2 \) as functions of \( r \), such that

\[
C_i' (r) J_i (nr) + C_i' (r) Y_i (nr) = 0,
\]

(13b)

\[
C_i' (r) J_i (nr) + C_i' (r) Y_i (nr) - f (r) = 0.
\]

(13c)

Solving equation (13a) and (13b) for \( C_i' (r) \) and \( C_i' (r) \) and integrating with respect to \( r \), we get

\[
f_i = \frac{S_1 J_i (m_1 r)}{n^2 - m_1^2} + \frac{S_2 J_i (m_2 r)}{n^2 - m_2^2}.
\]

(13d)

The general solution of (11a) for a cylinder \( 0 \leq r \leq a \) may be written as

\[
f_i (r) = C_i J_i (nr) + S_1 \frac{J_i (m_1 r)}{n^2 - m_1^2} + S_2 \frac{J_i (m_2 r)}{n^2 - m_2^2}.
\]

(14a)

From (14a) in equations (5) we get

\[
f_x (r) = \frac{1}{q} \left[ \begin{array}{c}
-nC_i J_i (nr) + \left( A_i - S_1 \frac{m_i}{n^2 - m_i^2} \right) J_i (m_1 r)
+ \left( A_i - S_2 \frac{m_i}{n^2 - m_i^2} \right) J_i (m_2 r)
\end{array} \right].
\]

(14b)
\[ f_s(r) = S_4 J_1(m_2 r) + S_5 J_0(m_2 r) - \frac{\rho_{12}}{\rho_{22}} \frac{n_1}{n_2} C J_1(nr). \]  

Where

\[ S_3 = \frac{m_1}{\rho_{22} w^2} [QA_1 + RB_1] - \frac{\rho_{12}}{\rho_{22}} \frac{nS_1}{m_2^2 - n^2}, \]

\[ S_4 = \frac{m_2}{\rho_{22} w^2} [QA_2 + RB_2] - \frac{\rho_{12}}{\rho_{22}} \frac{nS_2}{m_2^2 - n^2}, \]

\[ S_5 = \frac{1}{\rho_{22} w^2} [QA_1 + RB_1] - \frac{\rho_{12}}{\rho_{22}} \left[ S_1 \frac{m_1}{n^2 - m_1^2} - A_1 \right], \]

\[ S_6 = \frac{1}{\rho_{22} w^2} [QA_2 + RB_2] - \frac{\rho_{12}}{\rho_{22}} \left[ S_2 \frac{m_2}{n^2 - m_2^2} - A_2 \right]. \]

IV. BOUNDARY CONDITIONS

The Boundary conditions are:

\[ \tau_{rr} = \tau_{rr} = 0, \quad S = -S_0, \quad r = a. \]

The arbitrary integration constants \( A_1, A_2 \) and \( C_i \) are obtained from the boundary conditions (15) in the form

\[ x_{11} C_1 + x_{12} A_1 + x_{13} A_2 = 0, \]

\[ x_{21} C_1 + x_{22} A_1 + x_{23} A_2 = 0, \]

\[ x_{32} A_1 + x_{33} A_2 = -S_0. \]

Hence

\[ A_1 = \frac{x_{21} x_{13} - x_{11} x_{23}}{x_{12}} \left( x_{21} x_{13} - x_{11} x_{23} - x_{31} x_{21} - x_{11} x_{22} \right) S_0, \]

\[ A_2 = \frac{-x_{21} x_{12} - x_{11} x_{22}}{x_{12}} A_1, \]

\[ C_i = -\frac{x_{12} A_1 + x_{13} A_2}{x_{12}}. \]

V. NUMERICAL RESULTS AND DISCUSSIONS

For numerical calculations, we consider the following the non-dimensional variables

\[ \left( u_r, u_r', v_r', v_r', r', z' \right) = \frac{1}{a} \left( u_r, u_r', v_r', v_r', r^*, z^* \right), \]

\[ w = \frac{w}{w}, \quad w = \frac{\pi c_s}{\alpha}, \quad q = q \alpha, \quad t^* = tw, \]

\[ \left( \tau_{rr}, \tau_{r0}, \tau_{rr}, \tau_{rr}, S \right) = \frac{1}{S_0} \left( \tau_{rr}, \tau_{r0}, \tau_{rr}, \tau_{rr}, S \right). \]

We get the values of the non-dimensional radial displacement, the radial stress, the axial- shear stress, and the pore pressure as a function of radial distances for five values of porosity \( \beta \) (0.30, 0.32, 0.34, 0.36, 0.38) and when the non-dimensional frequency \( \omega \) is equal to 1. Figure 1 portrays the dimensionless radial displacement \( u_r \) versus the dimensionless radius \( r \) for five porosity \( \beta \) (0.30, 0.32, 0.34, 0.36, 0.38). It is observed that \( u_r \) first attains a maximum value and then decreases with distance, until eventually it equal to zero at the center of cylinder. The value of dimensionless radial displacement \( u_r \) increases with porosity \( \beta \) and the dimensionless radius \( r \). Figure 2 portrays the variation of radial stress \( \tau_{rr} \) versus the dimensionless radius \( r \) for several values of the porosity \( \beta \). The value of dimensionless radial stress \( \tau_{rr} \) increases with porosity \( \beta \) while decreases with dimensionless distance. Figure 3 show the dimensionless pore pressure \( S \) versus dimensionless distance. It is observed that \( S \) increases with...
porosity $\beta$ while decreases with dimensionless distance. Figure 4 illustrate the dimensionless axial-shear stress $\tau_{rz}$ and show that the axial-shear stress $\tau_{rz}$ decreases with $\tau$ at the value of $r = 0.5$ and then it starts to increase. It is observed that $\tau_{rz}$ decreases with porosity $\beta$. Figures 5, 6, 7 and 8 portrays the variation of $u_r$, $\tau_{rr}$, $S$ and $\tau_{rz}$ as a function of radial distances respectively, for five values of $S(1, 2, 3, 4, 5)$.

REFERENCES

Figure 2: Variation of radial stress $\tau_r$ with radial distance $r$ for five porosity $\beta$.

Figure 4: Variation of axial-shear stress $\tau_\alpha$ with radial distance $r$ for five porosity $\beta$.

Figure 3: Variation of pore pressure $S$ with radial distance for five porosity $\beta$.

Figure 5: Variation of radial displacement $u_r \times 10^{-11}$ with radial distance $r$ for five $S_0$. 

Ibrahim A. Abbas was born on November 20, 1971 in Sohag Egypt. In 2004, he received PhD in Mathematics at South valley University, Egypt. Member of the Egyptian Mathematical Society. At present, his affiliation is the Sohag University, Egypt. He works in the field of theory of thermoelasticity and fluid mechanics by finite element method. His more detailed CV can be found in “Who’s Who in Science and Engineering.”